A CRACKED COLUMN UNDER COMPRESSION

HIROYUKI OKAMURA, H. W. LIU and CHORNG-SHIN CHU
Syracuse University, Syracuse, N.Y. 13210, U.S.A.

and

H. LIEBOWITZ

Abstract - A crack reduces the flexural rigidity of a column. The effects of the reduced rigidity on the load carrying capacity, the deflection, and the fracture load of a slender column with a single-edge crack have been studied based on the column theory together with the well-known relationship between the compliance and the stress intensity factor of a cracked beam.

A crack reduces the load carrying capacity and increases the lateral deflection of a column under an eccentric compressive load. The calculated deflection agrees very well with the values measured by Liebowitz, Vanderveldt and Harris. The increased lateral deflection increases the bending moment at the cracked section. The bending stress intensity factor of a cracked column was also calculated, with the increased bending moment taken into consideration.

The net stress intensity factor at the crack tip is the difference between the bending and the compression stress intensity factors. The fracture toughness value obtained from cracked columns agree reasonably well with the value obtained from cracked plate. This study indicates that the superposition of stress intensity factors of the same mode is valid.

INTRODUCTION

The elastic stability of columns has been studied extensively[1]. Fracture of cracked beams under bending and fractures of cracked plates under tension have also been studied extensively in recent years[2]. However, stability and fracture of cracked columns under a compression load have been studied only very recently, such as experimental work by Liebowitz, Harris and Vanderveldt[3] and theoretical work by Liebowitz and Claus[4] on fracture of various notched columns utilizing the stress intensity factor and the Neuber stress concentration expressions. In the present study, the load carrying capacity, the deflection, as well as the fracture load of a slender column with a single-edge crack have been calculated, primarily using beam-column analysis and stress intensity values given by Srawley and Gross[5]. The flexural rigidity of a cracked column can be determined from the stress intensity factor given in [5] of a single-edge cracked beam under pure bending.

A crack may reduce the flexural rigidity of a column and its load carrying capacity. From the reduced flexural rigidity, the deflection and the load carrying capacity of a notched column may be calculated. The bending at the cracked section of a column causes a tensile mode crack tip stress field, which is characterized by a stress intensity factor. When the stress intensity factor at a crack tip exceeds the fracture toughness of the material, fracture occurs. The fracture toughness of cracked columns is studied.

CRACKED COLUMNS UNDER COMPRESSION—HINGED ENDS

Figure 1 shows a single-edge cracked column with hinged ends. The column has a rectangular cross-section with width and thickness $W$ and $B$ respectively. The eccentricity of the compression is $e$, and the deflection of the midsection is $\delta$. For 'concentric' loading, $e$ is zero. In this section, the deflection and the load carrying capacity of an elastic cracked column are calculated.
To facilitate the analysis, the column is divided into three sections, the center section of length $l^*$, which contains a crack of length $a$, and two outer sections without crack each of length $l$. The crack in the center section reduces the flexural rigidity of the column and its load carrying capacity.

Assuming that the eccentricity is in the plane of symmetry as shown in Fig. 1, the bending moment at any cross-section of the column is

$$M = P(y + e).$$  \hspace{2cm} (1)

The differential equation for beam deflection is then

$$EJ \frac{d^2 y}{dx^2} = -P(y + e).$$ \hspace{2cm} (2)

with the conditions that the moment and the slope of the column are continuous at the intersection of the center and the outer sections, and

$$y = 0 \text{  at  } x = 0.$$ \hspace{2cm} (3)

Let $\lambda^2 = P/EI$. Equation (2) is then

$$\frac{d^2 y}{dx^2} + \lambda^2 y = -\lambda^2 e.$$ \hspace{2cm} (4)

The general solution of (4) is

$$y = A \sin \lambda x + B \cos \lambda x - e.$$ \hspace{2cm} (5)
The end condition of \( y = 0 \) at \( x = 0 \) leads to \( B = e \). Therefore, the deflection is

\[
y = A \sin \lambda x + e (\cos \lambda x - 1).
\]

At the intersection of the outer and the center sections, the moment and the slope are continuous. The moment and the slope, \( M \) and \( \theta \), of the outer section are related to its lateral deflection. The moment and the slope, \( M^* \) and \( \theta^* \), of the center section are related to each other by the beam compliance of the cracked center section.

An eccentric compression load on a column can be resolved into a moment and a concentric compression as shown in Fig. 1c. The moment arm of the bending moment at the mid-section of the column is the sum of the eccentricity and the deflection i.e., \( (e + \delta) \). Therefore, the resolved moment is \( P(e + \delta) \) and the resolved concentric compression is \( P \). For an uncracked column, the moment gives a linear stress distribution across the width of the column as shown by ABC in Fig. 1d. The compression gives a uniform stress distribution \( DE \). Assume that the crack is at the left side of the column as shown. In such a case, the crack will affect the column only if the tensile stress caused by the bending moment exceeds the compressive stress caused by the load \( P \), i.e. \((\sigma_m + \sigma_P) > 0\). It can be shown easily that this is the case if the compression is located at \( W/6 \) to the right of the center line. Therefore, the eccentric compression can be resolved into a compression located at \( W/6 \) and a bending moment, \( \bar{M}^* = P(e + \delta - W/6) \) as shown in Fig. 2. This compression at \( W/6 \) alone does not open up the crack. As the moment \( \bar{M}^* \) is applied, the crack opens gradually, and the effect of reduced stiffness is realized. This bending moment at both ends of the center section.
and the relative rotation, $\theta^*$, between these two ends must follow the relation

$$\tilde{M}^* = k\theta^*$$  \hspace{1cm} (7)

for an elastic cracked beam, where $k$ is a rotational spring constant. The moment at the end of the center section is

$$M^* = \tilde{M}^* + PW/6.$$  \hspace{1cm} (8)

The continuity conditions of the moment and the slope at the intersection between the outer and the center sections lead to

$$EI\lambda^2 (A \sin \lambda l + e \cos \lambda l) = \tilde{M}^* + PW/6 = 2k\lambda (A \cos \lambda l - e \sin \lambda l) + PW/6$$  \hspace{1cm} (9)

noticing the fact that $\theta^* = 2\theta = 2dy/dx$. After the constant $A$ is evaluated from (9), the column deflection can be calculated as

$$y = \frac{\beta e \lambda^2 P \cos \lambda l + \lambda l e \sin \lambda l}{\frac{PW}{12k} \lambda l \cos \lambda l - \beta \lambda^2 P \sin \lambda l} \sin \lambda x + e \left( \cos \lambda x - 1 \right)$$  \hspace{1cm} (10)

where $\beta = EI/2kl$. If one assumes that the length of the center section is zero, i.e. $l^* = 0$, the deflection, $\delta$, at the mid-section of the column is given by

$$\frac{\delta}{e} = \frac{1 - \frac{W}{6e} \beta \lambda l \sin \lambda l}{\cos \lambda l - \beta \lambda l \sin \lambda l} - 1.$$  \hspace{1cm} (11)

The assumption $l^* = 0$ will be discussed in more detail later.

The quantity $\delta/e$ depends on the parameters, $\beta$, $\lambda l$, and $W/e$. The effect of a crack is contained in the parameter $\beta$. As will be shown later, for an uncracked column, $\beta = 0$. In this case, (11) agrees with the already known deflection of an uncracked column under eccentric compression[7]. For columns having small values of $W/e$, $\beta$, and $\lambda l$, the quantity $(W/\beta \lambda l \sin \lambda l)/6e$ in (11) can be neglected, such as for a narrow and short column with a short crack and a large eccentricity. In such a case

$$\frac{\delta}{e} = \frac{1}{\cos \lambda l - \beta \lambda l \sin \lambda l} - 1.$$  \hspace{1cm} (12)

When the relative rotation between the two ends of the center section is calculated, the moment at the end of the center section is reduced by an amount of $PW/6$ as shown by (7) and (8). This reduction causes the difference between the deflections as given by (11) and (12). The moment $M^*$ is reduced by the amount of $PW/6$, because of the compressive bearing stress between the upper and the lower crack surfaces.

As given by (12), the deflection, $\delta/e$, is a function of $\beta$ and $\lambda l$. In this case, the deflection can be plotted easily as shown in Fig. 3. It should be noted that (12) and Fig. 3 are applicable only if the quantity $(W/\beta \lambda l \sin \lambda l)/6e$ is negligible.

The curves in Fig. 3 indicate that as the load increases, the deflection, $\delta$, increases.
As the deflection approaches infinity, the load approaches an asymptotic value. The asymptotic value can be derived from (12), with the condition $\delta/e = \infty$. It is

$$\beta \lambda l \tan \lambda l = 1.$$  \hspace{1cm} (13)

This asymptotic load is the load carrying capacity of a cracked elastic column, if the column can sustain very large deformation. It is contained in the parameter ($\lambda l$). In Figs. 4 and 5, a plot of $(\lambda l)^2$ is shown as a function of $\beta$. The figure indicates that, as the crack length increases, the column becomes less rigid, and the load carrying capacity decreases.

Equation (13) indicates that the load carrying capacity of a cracked column is independent of eccentricity. In other words, (13) applies even if the eccentricity is vanishingly small. This is not correct, because the crack surfaces bear compressive stress until the location of the load application moves beyond $W/6$ from the center line of the cracked section. For a cracked column under concentric compression, the deflection remains zero until the buckling load of an uncracked column is reached. Therefore, the load carrying capacity of a cracked column under concentric compression is the same as the buckling load of an uncracked column.

When eccentricity is less than $W/6$, the load carrying capacity could be higher than that given by (13), because crack surfaces bear compressive stress. In such a case, the load carrying capacity can be found from (11) and (13) or from Figs. 3 and 4 or 5.

Suppose that for a cracked column, the eccentricity, $e$, is less than $W/6$, then the effect of the crack will not be realized until $(e + \delta) = W/6$. Therefore, the load-deflection relation of the column follows the curve of an uncracked column in Fig. 3, until $\delta/e = (W/6e - 1)$. If the applied load at this point is lower than the load carrying capacity of the column given by (13) or Fig. 4, the load-deflection curve shifts to that of a cracked column. If the applied load at this point is higher than that given by (13) or Fig. 4, the load at this point is effectively the load carrying capacity of the column.
In Fig. 6a, the ratio of $P/P_0$ is plotted as a function of $\beta$. $P$ is the load carrying capacity according to (13) and $P_0$ is the buckling load of an uncracked column under concentric compression. This is the same relation as in Fig. 4.

Figure 6b is obtained from the load-deflection curve of an uncracked column in Fig. 3. The curve gives the relationship between the load ratio $P/P_0$ and the eccentricity such that at the load level, $(e+\delta) = W/6$. Figures 6a and 6b can be used to find the load carrying capacity of a cracked column when $e < W/6$. For a given column, the value of $\beta$ is known. Assume $\beta = 0.2$. Construct a line $AB$ to intersect the curve in Fig. 6a. The ordinate of $B$ is the load carrying capacity of a cracked column as given by (13) for the case of a large eccentricity. The abscissa of $C$ gives the value of $e/W$ such that at the load level of $B$, $(e+\delta) = W/6$. In the case of $\beta = 0.2$, the value of $e/W = 0.042$. If $e/W < 0.042$, the load carrying capacity is given by the curve $EC$; if $e/W > 0.042$, the load carrying capacity is constant at $P/P_0 = 0.7$.

The above discussion is based on the fact that crack surfaces can bear compressive stress. If the crack is replaced by a slit, a concentric load will cause the mid-section of the cracked column of Fig. 1 to deflect to the right; i.e. in the direction away from the crack. In this case, the load carrying capacity of the column is predicted by (13).

When the deflection was calculated, the eccentric compression was resolved into a moment and a compression located at $W/6$ from the center line of the column as
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Fig. 5. Load carrying capacity of a cracked column.

Fig. 6. Load carrying capacity of a cracked column.
shown in Fig. 2. The effect of the resolved compression on the lateral deflection will now be discussed.

The increased lateral deflection of a cracked column is caused by the increased compliance of the cracked section. Compliance is defined as $C = 1/k$, where $k$ is a rotational spring constant. When the compression is applied near the center line of the column, the increased compliance of the cracked section is not realized until the location of the load moves to the right of $W/6$ from the center line, see Fig. 2. In the above calculation, the effect of the crack is assumed to be fully realized, after the load is moved beyond $W/6$ from the center line. In this case, the compliance, is shown by the solid line $ABCD$ in Fig. 6, as a function of the location of load application. This assumed compliance is approximately correct.

As the load moves to the right of $W/6$, the bending moment opens up the crack tip; however, the compression may keep the crack tip closed. The crack tip will fully open only when the magnitude of the bending stress intensity factor $K_m$ exceeds that of the compressive stress intensity factor, $K_p$, i.e.

$$K_m > K_p.$$  \hfill (14)

The stress intensity factors of both bending and axial load are given by Gross and Srawley [5]:

$$\frac{K_m B W^2}{6Ma^{1/2}} = Y_m = 1.99 - 2.47(\frac{a}{w}) + 12.97(\frac{a}{w})^2 - 23.17(\frac{a}{w})^3 + 24.80(\frac{a}{w})^4$$ \hfill (15)

and

$$\frac{K_p B W}{Pw^{1/2}} = Y_p = 1.99 - 0.41(\frac{a}{w}) + 18.70(\frac{a}{w})^2 - 38.48(\frac{a}{w})^3 + 53.85(\frac{a}{w})^4.$$ \hfill (16)

The condition of (14) leads to

$$(\sigma + \delta) > \frac{WY_p}{6Y_m}.$$ \hfill (17)

The crack tip is fully open when the load is moved to the right of $WY_p/6Y_m$. The crack is fully closed, when the load is applied to the left of $W/6$. When the load is located in between these two points, the crack surfaces near the tip are closed, and compressive bearing stress exists between the contacted crack surfaces. However, part of the crack surfaces, further away from the tip, remains separated. Therefore, within the region from $W/6$ to $WY_p/6Y_m$, the compliance of the cracked section is in between zero and that of a fully opened crack. This is shown schematically by the dashed line in Fig. 7.

The location of load application for a fully opened crack is determined by the ratio $Y_p/Y_m$. In Fig. 8, the ratio $Y_p/Y_m$ is plotted as a function of $a/W$. When the crack is very short, the ratio $Y_p/Y_m$ is nearly unity. In this case, the assumed compliance $ABCD$, see Fig. 7, is nearly correct. As the crack length increases, the assumption introduces a small error.

With (11, 12 and 13), the column deflection and the maximum load carrying capacity of a cracked column can be calculated, if the spring constant, $k$, or the compliance, $C = 1/k$ is known.
The spring constant $k$ can be obtained from the stress intensity factor of a cracked beam with the relationship between flexural compliance and the stress intensity factor, $K_m$. The stress intensity factor of a cracked beam is related to its compliance, $C$, by [6]

$$\frac{(1-\nu^2)K_m^2}{E} = \frac{M^2}{2B} \frac{dC}{da}$$

(18)

where $B$ is the beam thickness; $a$, crack length; and $\nu$ = Poisson's ratio for plane strain case, and $\nu = 0$ for plane stress case. This definition of $\nu$ is used throughout this paper unless it is stated otherwise. The stress intensity factor of a single-edge
cracked beam under pure bending is given by (15). Integrating (18), with $K_m$ as given by (15), one obtains

$$C - C_0 = \frac{72(1 - \nu^2)}{EBW^2} \int_0^{a/w} \frac{a}{w} \left[ Y_m \left( \frac{a}{w} \right) \right]^2 d \left( \frac{a}{w} \right) = \frac{72(1 - \nu^2)}{EBW^2} F \left( \frac{a}{w} \right)$$

(19)

where $C_0$ is the compliance of an uncracked beam. From (19) and with the relation $k = 1/C$, one derives the spring constant

$$k = \frac{1}{E \left[ \frac{l^*}{1 + \frac{72(1 - \nu^2)}{BW^2} F \left( \frac{a}{w} \right)} \right]}$$

(20)

where $l^*$ is the length of the center section and

$$F \left( \frac{a}{w} \right) = 1.98 \left( \frac{a}{w} \right)^2 - 3.277 \left( \frac{a}{w} \right)^3 + 14.43 \left( \frac{a}{w} \right)^4 - 31.26 \left( \frac{a}{w} \right)^5 + 63.56 \left( \frac{a}{w} \right)^6$$

$$- 103.36 \left( \frac{a}{w} \right)^7 + 147.52 \left( \frac{a}{w} \right)^8 - 127.69 \left( \frac{a}{w} \right)^9 + 61.50 \left( \frac{a}{w} \right)^{10}$$

(21)

The function $F(a/w)$ is plotted in Fig. 9. If $l^*$ is assumed to be zero, the spring constant is then

$$k = \frac{EBW^2}{72(1 - \nu^2) F \left( \frac{a}{w} \right)}$$

(22)

![Fig. 9. The function $F(a/W)$ of a single-edge cracked beam.](image-url)
When the lateral deflection was calculated, $I^*$ was assumed zero, see (11), so that the usual deflection of a smooth column of the center section has been accounted for; therefore, to calculate the deflection at the mid-section, only the additional rotation caused by the crack in the center section needs to be included. When the spring constant is calculated, $I^*$ was assumed to be zero also. By comparing (19) and (22), one notices that the assumption of $I^* = 0$ is equivalent to defining $k = 1/C - C_0$. $C$ and $C_0$ are compliances of the center section with and without a crack. When the crack length approaches zero, $C$ approaches $C_0$, $k$ to $\infty$, and $\beta$ to zero. Therefore, it is the difference between the compliances of a cracked and an uncracked column that contributes to the decrease of $k$, which increases the lateral deflection. This assumption of $I^* = 0$ neglects the usual deflection of a smooth column of the center section; only the deflection caused by the additional rotation of the center section is taken into consideration. The assumption of $I^* = 0$ for the calculations of both the deflection and the spring constant is equivalent to a mechanical model of a column of length $2l$, with a rotational spring at the mid-section having a spring constant given by (22).

With the spring constant known, the parameter $\beta$, can be calculated. Then the deflection at the mid-section of the column can be calculated with (11) or (12). Figures 10 and 11 show the comparison of the calculated deflection with the deflection measured by Liebowitz et al.[3]. The solid curves are calculated and the dashed ones are measured. A value of $10.5 \times 10^6$ psi is used for $E$, and 13.25 in. for $2l$. Consider the

![Fig. 10. Load-deflection curve for a column with a 0.09 in. deep notch with a fatigue crack extension.](image-url)
simple-minded model used for the calculation, the agreement between the calculated and the measured values are very good. The maximum deviation is approximately 5 per cent in load.

With the spring constant, $k$, known, the load carrying capacity of the column can also be calculated. Equations (13) and (22) lead to an expression for the critical load of a single-edge cracked column.

\[
\frac{1}{3(1-v^2)\frac{w}{l}F\left(\frac{a}{w}\right)} = \lambda l \tan \lambda l. \tag{23}
\]

The load carrying capacity, $P$, is contained in the parameter $\lambda l$ and it is related to two dimensionless quantities $(1-v^2)\frac{w}{l}$ and $a/w$.

Equation (23) can be solved numerically or graphically. Figure 12 shows the result of the solution to (23). The ratio of the load carrying capacity of a cracked column and the buckling load of an uncracked column ($P_{cr}/P_0$) is plotted against $a/w$ for six values of $(1-v^2)\frac{w}{l}$ such as: 0.01, 0.02, 0.04, 0.1, 0.2 and 0.4. The figure shows that a crack does reduce the load carrying capacity of a column. However, the effect of a crack decreases with both $a/w$ and $w/l$. For a very slender column or a column with a small crack, the critical load of the column is not affected much by the crack. For a
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A cracked column with a crack 0.6 w long and the column having a value of 0.01 for \(1 - \nu^2\) w/l, the load carrying capacity is reduced only 11 per cent from that of an uncracked column. The value of 0.01 for \(1 - \nu^2\) w/l corresponds to a slenderness ratio of 620. For a crack length \(a = 0.2 w\), the load carrying capacity is reduced only 8 per cent for \(1 - \nu^2\) w/l = 0.20, which corresponds to a slenderness ratio of 31. For longer crack and shorter column, the load carrying capacity decreases rapidly. Liebowitz et al.[3] measured the load carrying capacities of concentrically loaded single edge cracked columns of 12" long with various notch depths. For notch depth ratios \(a/w\) of 0.18, 0.34 and 0.39, the calculated ratios of \(P_{cr}/P_o\) are 0.98, 0.91 and 0.88 respectively. The reduction in load carrying capacity is not appreciable.

Eccentricity has an effect on the load carrying capacity of a cracked column, which has been discussed earlier in the paper. The above discussion is valid only for an elastic cracked column which can sustain a large deflection. Often the tensile crack tip stress field causes fracture before the load carrying capacity given by (13) and (23) is reached. Fracture of cracked column will be discussed in the next section.

**FRACTURE OF CRACKED COLUMN**

As shown in Fig. 3, the lateral deflection at the mid-section, \(\delta\), increases slowly with the load at first, then rapidly after the applied compression passes a certain value. As both the deflection and the compression increase, the applied bending moment at the cracked-section increases rapidly. This bending induces tensile mode, i.e., Mode I, crack tip stress field. If the material is brittle, the column may fracture before the load carrying capacity is reached.

The recent developments of linear elastic fracture mechanics are well known[2]. It has been substantiated that when the stress intensity factor at a crack tip reaches the fracture toughness of a brittle material, the cracked member fractures.
The stress intensity factor of a beam under bending is given by (15). The moment, $M$, at the cracked section of the column is

$$M = Pe\left(\frac{\delta}{e} + 1\right).$$  \hfill (24)

With (11) for $\delta/e$, the moment at the mid-section is

$$M = \frac{Pe\left(1 - \frac{w}{6e} \beta l \sin l\right)}{\cos l - \beta l \sin l}.  \hfill (25)$$

Substituting into and re-arranging (18), one obtains

$$K_m = \frac{1}{2} \left(\frac{w}{l}\right)^2 \frac{e}{w^2/2} \frac{1 - \frac{w}{6e} \beta l \sin l}{\cos l - \beta l \sin l} \left(\frac{a}{w}\right)^{1/2} Y_m.  \hfill (26)$$

The function $Y_m$ is given by (15).

When the quantity $(W/6e) \beta l \sin l$ is negligible, (26) can be rearranged as

$$K_m W^{1/2} = \frac{1}{2} \left(\frac{w}{l}\right)^2 \frac{e}{w^2/2} \frac{\lambda^{1/2}}{\cos l - \beta l \sin l} \left(\frac{a}{w}\right)^{1/2} Y_m.  \hfill (27)$$

According to (26), the stress intensity factor of a single-edge cracked column under an eccentric compression is related to five dimensionless parameters. The parameter, $\lambda l = (P/EI)^{1/2}$, is the usual column parameter which compares the rigidity of the column to the load. The parameter $\beta = EI/2kl$ contains the effect of a crack. It compares the flexural rigidity of a cracked column with that of an uncracked one.

The function $\{(a/w)^{1/2} Y_m (a/w)\}$ is plotted against $a/w$ in Fig. 13. With (26) or (27), the bending stress intensity factor, $K_m$, of a cracked column can be calculated. However the net stress intensity factor at the crack tip is reduced by compression, and it is

$$K = K_m - K_p = K_m \left\{1 - \frac{W/6}{(\delta + e) Y_m}\right\}  \hfill (28)$$

where $K_p$ is the stress intensity factor due to axial compression, and it can be calculated from (16). The ratio of $Y_p/Y_m$ is plotted in Fig. 8. Equation (28) indicates that when the moment arm increases, the effect of the compression load decreases. When the moment arm is short the effect of compression is fairly large.

Figure 14 shows the calculated load-deflection curves of the specimens, which have been shown previously in Figs. 10 and 11. However, in Fig. 14, the quantity $(e+\delta)$ is plotted rather than $\delta$ alone. The experimental fracture points are transposed onto the calculated curves.

In Fig. 14, ten specimens are shown with various eccentricities and two crack lengths. The solid and dashed load-deflection curves are for crack lengths $0.09$ and $0.195$ in. respectively. The fracture points are marked by crosses. The constant fracture toughness lines for specimens having the same crack length are calculated from (28) and shown by dash-dotted lines.
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Fig. 13. \((a/w)^{1/2}Y_m(a/w)\) as a function of \(a/w\).

Fig. 14. Fracture toughness of cracked column.
The fracture toughness value of $37 \times 10^3$ psi(in.)$^{1/2}$ correlates very well with experimental fracture points of specimens having a crack 0.09 in. long, and the value of $30 \times 10^3$ psi(in.)$^{1/2}$ with specimens having a crack 0.195 in. long. These two values are reasonable in comparison with $37 \times 10^3$ psi(in.)$^{1/2}$ as measured for plate specimens of 0.75 in. thick [8]. The thickness of the columns used by Liebowitz et al. is 0.5 in.

In this diagram, contours of constant bending stress intensity factor, $K_m$, are lines having a slope of $-1$ as shown by the dotted lines. These two lines are for constant $K_m$-values of $37 \times 10^3$ psi(in.)$^{1/2}$ and $30 \times 10^3$ psi(in.)$^{1/2}$ for specimens having crack lengths of 0.09 in. and 0.195 in. respectively. The difference between the dash-dotted and the dotted lines is the effect of the compression load.

It is indicated clearly that (28) correlates very well all the fracture points of specimens having the same crack length. However, the specimens with a shorter crack have a higher fracture toughness value.

It is appropriate to point out that the fracture mechanics is applicable only if fracture is rather 'brittle'. The important requirement for brittle fracture is that the specimen size is sufficiently large relative to the crack tip plastic zone size at fracture [9-11]. Experimental evidences indicate that if the specimen size is not large enough, a specimen with a shorter crack gives a higher toughness value [11]. This qualitative trend agrees with the data shown in Fig. 14.

Fracture toughness calculated from (28) takes both bending and compression stress intensity factors into consideration. The calculated and the experimental results correlate very well. This, therefore, indicates the validity of the superposition of stress intensity factors of the same mode.

**SUMMARY AND CONCLUSIONS**

1. The lateral deflection, the load carrying capacity and the stress intensity factor of a cracked column under compression were calculated using the reduced flexural rigidity of a cracked column.
2. A crack reduces the flexural rigidity and the load carrying capacity of a column under an eccentric compression.
3. The buckling loads of cracked and uncracked columns are the same, because the crack surfaces bear compressive stress.
4. The lateral deflection of a column under eccentric compression is increased by a crack. The calculated deflection agrees very well with the values measured by Liebowitz, Vanderveldt, and Harris.
5. The effects of a crack on load carrying capacity and lateral deflection, decreases with the crack length to column width ratio ($a/w$), and the column width to column length ratio ($w/l$). In other words, the effect is small, if a crack is short and a column is long.
6. The bending moment at a cracked section increases rapidly as both the applied eccentric compression and the lateral deflection increase. The bending moment causes tensile, Mode I, crack tip stress field. The bending stress intensity factor of a cracked column was calculated with Gross and Srawley's formula together with the lateral deflection at the cracked section.
7. The net stress intensity factor at a crack tip is the difference between the bending stress intensity factor and the compression stress intensity factor. When the moment arm is small, the effect of compression is very large. The effect decreases as the moment arm increases.
8. The fracture test conducted by Liebowitz et al. indicate that all the cracked columns having the same crack length yield nearly the same fracture toughness value. The fracture toughmesses measured from cracked columns agree reasonably well with the existing value.

9. This investigation indicates that the superposition of stress intensity factors of the same mode is valid.

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REFERENCES


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