Numerical analysis of two dimensional pin fins with non-constant base heat flux

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Abstract

This paper deals with the transient analysis of a two dimensional pin fin that contains tip convection and is subjected to time and space dependent heat flux at the fin base. A method of combining the Laplace transformation and finite difference is applied to analyze the problem. The general solution of the governing equation is first solved in the transform domain. The numerical transient solutions of temperature increments in the real domain are then obtained by using the matrix similarity transformation and Fourier series technique. Comparing with existing analytical solutions, the results presented here show high accuracy. The computational procedures established in this paper are also applicable to other types of boundary conditions, such as variable convection heat transfer coefficient.

Keywords: Laplace transformation; Finite difference; Fourier series technique; Pin fin

1. Introduction

Fins are widely used to increase the surface area and, consequently, to enhance the rate of heat exchange between a heated surface and a colder ambient fluid, such as in applications of

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semiconductors, heat exchangers, power generators and electronic components. So far, there have been many studies [1–3] focused on steady state analysis of one or two dimensional fins. The results of steady state analysis are suitable for fin design in most real applications. Nevertheless, the analysis of the transient response of fins is also important in some applications [4–9]. For convenience of analysis, most studies in the literature assumed a step change in the base temperature or in the base heat flux, and the others considered periodic cases. However, in reality, the temperature, or the heat flux, at the fin base might be a function of both time and space simultaneously.

The applications of short pin fins are usually found in compact heat exchangers [10] and air cooled turbine blades [11]. So far, as mentioned above, there have been many investigations dealing with the transient analysis of two dimensional pin fins. However, little of the literature is concerned with the transient analysis of a two dimensional pin fin that is subjected to a time and space dependent heat flux at the fin base. In this article, a hybrid numerical method is adopted to investigate the transient response of a two dimensional pin fin that contains tip convection and is subjected to a time and space variation base heat flux. The general solution of the governing equation is first solved in the transform domain by the method of combining the Laplace transformation and finite difference. Then, the inverse transformation to the real domain is completed by the method of matrix similarity transformation and Fourier series technique [12,13]. Comparing with existing analytical solutions, the results presented by the computational procedures of this paper are accurate and efficient. In addition, the presented procedures allow us to calculate the temperature values of specific nodes at a specific time and, therefore, are more economical.

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2. Analysis

The configuration of a two dimensional pin fin is shown in Fig. 1. The analysis is based on the following assumptions:

(a) The pin fin is initially at ambient temperature, \( T_\infty \).
(b) The material properties of the pin fin, the heat transfer coefficients of the ambient fluid \( h \) and \( h_T \) and the temperature of the ambient fluid \( T_\infty \) are all assumed to be constant.
(c) The temperature distribution varies with \( x^* \), \( r^* \), and \( t^* \) in the two dimensional pin fin.
(d) There is no heat source or sink in the pin fin.
(e) Radiation is neglected in the pin fin.

The governing differential equation for the temperature distribution in the pin fin and its associated initial and boundary conditions are

\[
\frac{\partial \theta^*(x^*, r^*, t^*)}{\partial t^*} = \alpha \left[ \frac{\partial^2 \theta^*(x^*, r^*, t^*)}{\partial x^{*2}} + \frac{1}{r^*} \frac{\partial \theta^*(x^*, r^*, t^*)}{\partial r^*} + \frac{\partial^2 \theta^*(x^*, r^*, t^*)}{\partial r^{*2}} \right] \tag{1}
\]

\[ t^* = 0 \quad \theta^*(x^*, r^*, 0) = T_\infty \tag{2} \]

\[ t^* > 0 \quad x^* = 0 \quad -k \frac{\partial \theta^*(0, r^*, t^*)}{\partial x^*} = q_0 \cdot f^*(r^*, t^*) \tag{3} \]

\[ x^* = L \quad k \frac{\partial \theta^*(L, r^*, t^*)}{\partial x^*} + h_T \cdot \left[ \theta^*(L, r^*, t^*) - T_\infty \right] = 0 \tag{4} \]

\[ r^* = 0 \quad \frac{\partial \theta^*(x^*, 0, t^*)}{\partial r^*} = 0 \tag{5} \]

Fig. 1. Configuration of a pin fin.
where \( q_0^* \cdot f^*(r^*, t^*) \) is the heat flux at the fin base.

Define the following non-dimensional variables:

\[
\begin{align*}
  r &= r^*/L \\
  x &= x^*/L \\
  t &= \alpha t^*/L^2 \\
  \theta &= (0^* - T_\infty^*)/(q_0^* \cdot L/k) \\
  \text{Bi}_a &= hR^*/k \\
  \text{Bi}_T &= hTL/k \\
  G &= L/R^* \\
  R &= R^*/L
\end{align*}
\]  

Substitute the non-dimensional quantities given in Eq. (7) into Eqs. (1)–(6), then the governing equation and its associated initial and boundary conditions have the following dimensionless forms:

\[
\begin{align*}
  \frac{\partial \theta(x, r, t)}{\partial t} &= \frac{\partial^2 \theta(x, r, t)}{\partial x^2} + \frac{1}{r} \frac{\partial \theta(x, r, t)}{\partial r} + \frac{\partial^2 \theta(x, r, t)}{\partial r^2} \\
  t &= 0 \quad \theta(x, r, 0) = 0 \\
  t > 0 \quad x = 0 \quad -\frac{\partial \theta(r, 0, t)}{\partial x} = f(r, t) \\
  x = 1 \quad \frac{\partial \theta(r, 1, t)}{\partial r} + \text{Bi}_T \cdot \theta(r, 1, t) = 0 \\
  r &= 0 \quad \frac{\partial \theta(0, x, t)}{\partial r} = 0 \\
  r &= R \quad \frac{\partial \theta(R, x, t)}{\partial r} + G \cdot \text{Bi}_a \cdot \theta(R, x, t) = 0 \\
  x &= 1 \quad \frac{\partial \theta(1, r, s)}{\partial x} + \text{Bi}_T \cdot \theta(1, r, s) = 0
\end{align*}
\]  

In order to study the effects of the material parameters on the heat transfer of the fin, the tip Biot number \( \text{Bi}_T \) can be rewritten as follows:

\[
\text{Bi}_T = \frac{hTL}{k} = \frac{hTL}{h} \cdot \frac{L}{R^*} \cdot \frac{hR^*}{k} = H \cdot G \cdot \text{Bi}_a
\]  

where \( H = hTL/h \) represents the ratio of the tip convective heat transfer coefficient to the lateral convective heat transfer coefficient.

Applying the Laplace transformation with respect to time to the problem of Eqs. (8)–(13), we get the following results:

\[
\begin{align*}
  s \cdot \bar{\theta}(x, r, s) &= \frac{\partial^2 \bar{\theta}(x, r, s)}{\partial x^2} + \frac{1}{r} \cdot \frac{\partial \bar{\theta}(x, r, s)}{\partial r} + \frac{\partial^2 \bar{\theta}(x, r, s)}{\partial r^2} \\
  x &= 0 \quad -\frac{\partial \bar{\theta}(0, r, s)}{\partial x} = \bar{f}(r, s) \\
  x &= 1 \quad \frac{\partial \bar{\theta}(1, r, s)}{\partial x} + \text{Bi}_T \cdot \bar{\theta}(1, r, s) = 0
\end{align*}
\]
Then, applying the central finite difference in Eq. (15), we obtain the following discretized equation for \( r \neq 0 \):

\[
s \cdot \tilde{\theta}_{i,j} = \frac{\tilde{\theta}_{i+1,j} - 2 \cdot \tilde{\theta}_{i,j} + \tilde{\theta}_{i-1,j}}{\Delta x_i^2} + \frac{1}{r_j} \cdot \frac{\tilde{\theta}_{i,j+1} - \tilde{\theta}_{i,j-1}}{2 \cdot \Delta r_j^2} - \frac{1}{r_j^2} \cdot \frac{\tilde{\theta}_{i,j+1} - 2 \cdot \tilde{\theta}_{i,j} + \tilde{\theta}_{i,j-1}}{\Delta r_j^2}
\]

where \( \Delta x_i = 1/(m - 1) \) and \( \Delta r_j = R/(n-1) \). It has to be noted that the term involving \( 1/r \) in Eq. (15) is not applicable when the denominator \( r \) is equal to zero. By L’Hospital’s rule, we can replace such term involving \( 1/r \) and get the following equation for \( r = 0 \):

\[
s \cdot \tilde{\theta}(x, r, s) = \frac{\partial^2 \tilde{\theta}(x, r, s)}{\partial x^2} + 2 \cdot \frac{\partial^2 \tilde{\theta}(x, r, s)}{\partial r^2}
\]

The discretized form of Eq. (21) is

\[
s \cdot \tilde{\theta}_{i,j} = \frac{\tilde{\theta}_{i+1,j} - 2 \cdot \tilde{\theta}_{i,j} + \tilde{\theta}_{i-1,j}}{\Delta x_i^2} + 2 \cdot \frac{\tilde{\theta}_{i,j+1} - 2 \cdot \tilde{\theta}_{i,j} + \tilde{\theta}_{i,j-1}}{\Delta r_j^2}
\]

Finally, Eqs. (20) and (22) can be rewritten as

\[
A_{i,j} \cdot \tilde{\theta}_{i-1,j} + (B_{i,j} - s) \cdot \tilde{\theta}_{i,j} + C_{i,j} \cdot \tilde{\theta}_{i+1,j} + D_{i,j} \cdot \tilde{\theta}_{i,j-1} + E_{i,j} \cdot \tilde{\theta}_{i,j+1} = 0
\]

for \( i = 1, 2, \ldots, m \) and \( j = 1, 2, \ldots, n \)

where for \( r = 0 \) (\( j = 1 \)):

\[
A_{i,j} = C_{i,j} = \frac{1}{\Delta x_i^2}, \quad B_{i,j} = -\frac{2}{\Delta x_i^2} - \frac{4}{\Delta r_j^2}, \quad D_{i,j} = E_{i,j} = \frac{2}{\Delta r_j^2}
\]

while for \( r \neq 0 \) (\( j \neq 1 \)):

\[
A_{i,j} = C_{i,j} = \frac{1}{\Delta x_i^2}, \quad B_{i,j} = -\frac{2}{\Delta x_i^2} - \frac{2}{\Delta r_j^2}, \quad D_{i,j} = \frac{1}{\Delta r_j} \cdot \left( 2 - \frac{\Delta r_j}{r_j} \right)
\]

\[
E_{i,j} = \frac{1}{2 \cdot \Delta r_j^2} \cdot \left( 2 + \frac{\Delta r_j}{r_j} \right)
\]

Substituting the boundary conditions of Eqs. (16)–(19) into Eq. (23) and writing it in matrix form, we obtain the following equation:
a numerical method. However, for the sake of saving computing time, Eq. (28) does not allow us
main by Eq. (28). Then, the inverse Laplace transformation to the time domain can be achieved by
m possesses a set of \( \{P\}_{i=1}^{n} \) terms. In general, we can obtain the solutions of all the
specifically to calculate some nodal values. Therefore, sometimes, it is not economical to solve for
\( m \) terms. In general, we can obtain the solutions of all the

\[
\begin{bmatrix}
B_{1,1} & 2A_{1,1} & 0 & \cdots & 2D_{1,1} & 0 & \cdots & 0 \\
A_{1,1} & A_{1,1} & 0 & \cdots & \cdot & \cdot & \cdot & \cdot \\
0 & A_{1,1} & B_{1,1} & \cdot & \cdot & E_{i,j} & \cdot & \cdot \\
0 & \cdot & 0 & \cdots & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & 0 & \cdots & A_{m-1,n} & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & \cdots & 2A_{m,n} & B_{m,n} - \eta_{xy}
\end{bmatrix}
\]

\(- s[I]\) \times

\[
\begin{bmatrix}
\bar{\theta}_{1,1} \\
\bar{\theta}_{2,1} \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\bar{\theta}_{m-1,n} \\
\bar{\theta}_{m,n}
\end{bmatrix}
= \begin{bmatrix}
\bar{Q}_{1,1} \\
0 \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\bar{Q}_{1,j} \\
\bar{Q}_{1,n}
\end{bmatrix}
\] (26)

where \( \eta_{xy} = 2 \cdot E_{m,n} \cdot \Delta r_n \cdot G \cdot Bi_a + 2 \cdot A_{m,n} \cdot \Delta x_m \cdot Bi_T \) and

\[
\bar{Q}_{1,j} = -2 \cdot A_{1,j} \cdot \Delta x_1 \cdot f_j (r, s)
\] (27)

Eq. (26) can be rewritten in the following form:

\[
\{[M] - s[I]\} \cdot \{\bar{\theta}_{i,j}\} = \{\bar{Q}_{i,j}\}
\] (28)

where the matrices \([M] \), \( \{\bar{\theta}_{i,j}\} \) and \( \{\bar{Q}_{i,j}\} \) are the corresponding matrices in Eq. (26). In Eq. (28),
\([M] \) is a \((m \times n) \times (m \times n) \) band matrix with real number, \( \{\bar{\theta}_{i,j}\} \) is a \((m \times n) \times 1 \) vector representing
the unknown temperatures and \( \{\bar{Q}_{i,j}\} \) is a \((m \times n) \times 1 \) complex vector representing the forcing
terms. In general, we can obtain the solutions of all the \( m \times n \) nodal points in the transform
domain by Eq. (28). Then, the inverse Laplace transformation to the time domain can be achieved by
a numerical method. However, for the sake of saving computing time, Eq. (28) does not allow us
specifically to calculate some nodal values. Therefore, sometimes, it is not economical to solve for
the nodal values of vector \( \{\bar{\theta}_{i,j}\} \) through Eq. (28), especially when we want to compute only some
nodal values. In the situation when only specific nodal values at a specific time are required, we
can calculate by matrix operation to save the huge computer time.

Since the \((m \times n) \times (m \times n) \) matrix \([M] \) is a nonsingular real matrix, it is diagonalizable and
possesses a set of \((m \times n) \) linearly independent eigenvectors. There exists a nonsingular transition
matrix \([P] \) such that \([P]^{-1}[M][P] = \text{diag}[M] \) and the matrices \([M] \) and \( \text{diag}[M] \) are similar, having
the same eigenvalues. The matrix \( \text{diag}[M] \) is defined as

\[
\text{diag}[M] =
\begin{bmatrix}
\lambda_1 & \cdot & \cdot & \cdot \\
\cdot & \lambda_2 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \lambda_{m \times n}
\end{bmatrix}
\] (29)

where \( \lambda_l (l = 1, 2, \ldots, m \times n) \) is the eigenvalues of matrix \([M] \).

Substituting Eq. (29) into Eq. (28), we obtain the following equation:

\[
\{[P]^{-1}[M][P] - s[P]^{-1}[I][P]\} \cdot [P]^{-1}\{\bar{\theta}_{i,j}\} = [P]^{-1}\{\bar{Q}_{i,j}\}
\] (30)

Eq. (30) can be rewritten as

\[
\{\text{diag}[M] - s[I]\} \cdot \{\bar{\theta}^*_i,j\} = \{\bar{Q}_{i,j}\}
\] (31)

where

\[
\{\bar{\theta}^*_i,j\} = [P]^{-1}\{\bar{\theta}_{i,j}\}
\] (32)
\{\overline{Q}_{ij}\} = [P]^{-1} \{\overline{Q}_{ij}\} \tag{33}

From Eq. (31), the following solutions can be obtained:
\[ \tilde{\theta}_{ij}^* = \frac{\overline{Q}_{ij}}{\lambda_i - s} \tag{34} \]

By applying the inverse Laplace transformation to Eq. (34), we get the solution \(\theta_{ij}^*\). However, sometimes, it is difficult to find the inverse Laplace transformation of the complicated Eq. (34) by the residue theorem. In this article, the inverse Laplace transformation is completed by a general method, known as the Fourier series technique (refer to Appendix A). Finally, after we have obtained \(\theta_{ij}^*\), we can calculate the solution \(\theta_{ij}\) by
\[ \{\theta_{ij}\} = [P] \{\theta_{ij}^*\} \tag{35} \]

3. Results and discussions

The foregoing theoretical analysis for a pin fin will be illustrated by the following numerical results in Figs. 2–6. First, to show the accuracy of our analysis, Fig. 2 compares the results by the presented method with those of analytical solutions in Ref. [10] for a pin fin that is subjected to a time and space variation base temperature change in such a function as \((1 - 0.1r^2)(0.9 + 0.1 \cos(2\pi t))\). In Fig. 2, it is found that the fin surface temperature distributions by those two methods are in good agreement.

Next, Figs. 3–6 show the numerical results of a pin fin that is subjected to a time and space dependent heat flux at the base in such a function as \(f(r,t) = e^{-\Omega t}(1 - 0.4r^2)\) in Eq. (10).
Theoretically, the performance of a two dimensional pin fin depends on the lateral Biot number $B_i_a$ the geometric parameter $G$ and the ratio of convective heat transfer coefficients $H$. Fig. 3(a) and (b) show the distributions of the temperature increments at the fin centerline ($r = 0$) and at the fin surface ($r = R$) as a function of $x$ and $t$ for the cases of $G = 5$, $H = 1$, $\Omega = 0.5$ and (a) $B_{i_a} = 0.2$, (b) $B_{i_a} = 0.5$, respectively. It can be found that the temperature increments decrease with increasing radial distance and the effect of 2D conduction is not negligible, especially at large values of time. Moreover, as $B_{i_a}$ increases, the 2D effect becomes more pronounced.

Fig. 4 illustrates the effects of time $t$ and the ratio of convective heat transfer coefficients $H$ on the distributions of the centerline temperature increments for given $B_{i_a} = 0.2$, $G = 5$ and $\Omega = 0.5$. It is shown that the variations of the distributions are insensitive to changes in $H$ at small values of
This is because all the energy transfer at the base will increase the internal energy of the fin at the beginning, and the energy has not yet penetrated from the base to the tip at small time. In addition, the effect of $H$ increases with increasing radial distance. In other words, the effect of $H$ is more significant near the tip of the pin fin. This phenomenon is reasonable according to the definition of $H$. Fig. 4 shows the effects of the geometry parameter $G$ and the ratio of convective heat transfer coefficients $H$ on the distributions of the centerline temperature increments for given $Bi_a = 0.2$, $G = 5$ and $\Omega = 0.5$. It is found that the effect of $H$ on the distributions is less significant as $G$ increases. The reason for this phenomenon is that, for a fixed value of $G$, more energy will be convected from the tip surface to the surroundings as $H$ becomes larger. Therefore, the temperature increments at the fin centerline decrease when $H$ increases.

Fig. 4. Effects of $H$ on centerline temperature distributions at various $t$ values with $Bi_a = 0.2$, $G = 5$ and $\Omega = 0.5$.

Fig. 5. Effects of $G$ and $H$ on centerline temperature distributions at $t = 1$ with $Bi_a = 0.2$ and $\Omega = 0.5$. 
Finally, Fig. 6 depicts the temperature increment distributions of the pin fin for various values of $X$ with $B_i a = 0.2$, $G = 5$, $H = 1$ and $t = 1$. Since, from the definition of $X$, a large $X$ value means that the heat flux of the exponential function will decay rapidly, the temperature increments of the fin decrease as $X$ increases in Fig. 6. In addition, the 2D effect decreases with the increase of $X$.

4. Conclusions

A new numerical method of combining the Laplace transformation and the finite difference is applied to analyze the transient problem of a two dimensional pin fin that contains tip convection and is subjected to a time and space dependent heat flux at the fin base. The accuracy of the presented method is checked by existing analytical solutions. Other types of boundary conditions, such as a time dependent change in base temperature, can also be solved by the method.

The numerical results show that the effect of 2D conduction is more significant at large time. As lateral Biot number $B_i a$ increases, the 2D effect becomes more pronounced. On the other hand, the variations of the temperature increment distributions are insensitive to changes in $H$ at small values of time for given $B_i a = 0.2$, $G = 1$ and $\Omega = 0.5$. In addition, the effect of the ratio of convective heat transfer coefficients $H$ on the distributions of temperature increments increases as $G$ decreases for given $B_i a = 0.2$, $t = 1$ and $\Omega = 0.5$.

Appendix A

In this paper, a general method, known as the Fourier series technique [12,13], is employed to complete the inverse Laplace transformation. When a function $F(t)$ is given, the Laplace transformation and its inversion formula are defined as
\[ F(s) = \int_{0}^{\infty} F(t) e^{-st} \, dt \]  
(A.1)

\[ F(t) = \frac{1}{2\pi} \int_{c-i\infty}^{c+i\infty} F(s) e^{st} \, ds \]  
(A.2)

where the selected, positive arbitrary constant, \( c \), should be greater than the real parts of all singularities of \( F(s) \).

By letting \( s = c + i\omega \) Eqs. (A.1) and (A.2) are converted to those of the Fourier transformation

\[ \mathcal{F}(c + i\omega) = \int_{0}^{\infty} F(t) e^{-ct} e^{-i\omega t} \, dt \]  
(A.3)

\[ F(t) = \frac{e^{ct}}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}(c + i\omega) e^{i\omega t} \, d\omega \]  
(A.4)

The above integral can be approximated by

\[ F(t) \approx \frac{e^{ct}}{2\pi} \sum_{n=-\infty}^{\infty} \mathcal{F}(c + in\Delta\omega) e^{in\Delta\omega} \]  
(A.5)

and letting \( \Delta\omega = \pi/T^* \), yields

\[ F(t) \approx \frac{e^{ct}}{2\pi} \sum_{n=-\infty}^{\infty} \mathcal{F}(c + in\pi/T^*) e^{in(\pi/T^*)t} \]  
(A.6)

Considering the relation

\[ \mathcal{F}(c + in\pi/T^*) e^{in(\pi/T^*)t} + \mathcal{F}(c - in\pi/T^*) e^{-in(\pi/T^*)t} = 2\text{Re}[\mathcal{F}(c + in\pi/T^*) e^{in(\pi/T^*)t}] \]  
(A.7)

and truncating the infinite series at the \( N \)th term, we have the following numerical formula:

\[ F(t) \approx \frac{e^{ct}}{T^*} \left[ \frac{1}{2} \mathcal{F}(c) + \text{Re} \sum_{n=1}^{N} \mathcal{F}(c + in\pi/T^*) e^{in(\pi/T^*)t} \right] , \quad 0 \leq t \leq 2T^* \]  
(A.8)

From this equation, the numerical value of \( F(t) \) at any time \( t \) in the interval \( 0 \leq t \leq 2T^* \) can be obtained.

At \( t = T^* \), Eq. (A.8) becomes

\[ F(T^*) \approx \frac{e^{cT^*}}{T^*} \left[ \frac{1}{2} \mathcal{F}(c) + \text{Re} \sum_{n=1}^{N} \mathcal{F}(c + in\pi/T^*) (-1)^n \right] \]  
(A.9)

from Eq. (A.9), the numerical value at the time \( T^* \) can be calculated.

References