Abstract

Using Hamilton’s principle derives the governing differential equations for the coupled bending–bending vibration of a rotating pretwisted beam with an elastically restrained root and a tip mass, subjected to the external transverse forces and rotating at a constant angular velocity. Using the mode expansion method derives the closed-form solutions of the dynamic and static systems. The orthogonal condition for the eigenfunctions of the system with elastic boundary conditions is discovered. The self-adjointness of the system is proved. Moreover, the Green functions of the system are obtained. The symmetric properties of the Green functions are revealed. The frequency response on the steady response of the beam is also investigated.

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1. Introduction

Rotating pretwisted beams, which have been used in a lot of mechanical applications such as turbine blades, helicopter rotor blades, and electric fan. Most of researches of the vibration problems of rotating pretwisted beams had been studied by using numerical method because of its complexity. No analytical solution for the vibration of a rotating pretwisted beam had been presented.

For the nonrotating pretwisted beam, approximation methods are very useful tools to investigate the free vibrations of the pretwisted beams, where exact solutions are difficult to obtain. Dawson [1] and Dawson and Carnegie [2] used the Rayleigh–Ritz method and transformation techniques to study the effects of uniform pretwist on the frequencies of cantilever blades. Carnegie and Thomas [3] and Rao [4–6] used the Rayleigh–Ritz method and Ritz–Galerkin method to study the effects of uniform pretwist and the taper ratio on the frequencies of cantilever blades, respectively. Gupta and Rao [7] and Abbas [8] used the finite element method to determine the natural frequencies
Nomenclature

\[ A(x) \] cross-sectional area of the beam

\[ B_{ij}(\xi) \] dimensionless bending rigidity, \( E(x)I_{ij}(x)/[E(0)I_{yy}(0)] \), \( i, j = x, y \)

\[ E(x) \] Young’s modulus of the material

\[ I_{ij} \] area moment inertia of the beam

\[ J_{yy}, J_{zz} \] the rotatory inertia of the tip mass about the \( Y \) and \( Z \) axis, respectively

\[ J_{yy}, J_{zz} \] the rotatory inertia of the tip mass about the \( y \) and \( z \) axis, respectively

\[ K_{yT}, K_{y0}, K_{zT}, K_{z0} \] translational and rotational spring constants at the root of beam in the \( y \) and \( z \) directions, respectively

\[ k_1, k_2 \] the parameters of the rotatory inertia of the tip mass

\[ L \] length of the beam

\[ M^* \] the tip mass attached at the free end of the beam

\[ m(\xi) \] dimensionless mass of the beam, \( \rho(x)A(x)/[\rho(0)A(0)] \)

\[ \tilde{m}(\xi) \] dimensionless centrifugal force, \( x^2 \int_{\xi}^{1} m(\xi)(r + \xi) d\xi + \delta_{3} x^2(1 + r) \)

\[ P(\xi, \tau) \] dimensionless external transverse force in the \( y \) direction, \( p(x, t)L^3/[E(0)I_{yy}(0)] \)

\[ p(x, t) \] external transverse force in the \( y \) direction

\[ Q(x, t) \] dimensionless external transverse force in the \( y \) direction, \( q(x, t)L^3/[E(0)I_{yy}(0)] \)

\[ q(x, t) \] external transverse force in the \( z \) direction

\[ R \] the radius of the hub

\[ r \] dimensionless radius of the hub, \( R/L \)

\[ \tilde{T} \] kinetic energy

\[ t \] time variable

\[ \bar{U} \] potential energy

\[ u(x, t) \] beam lateral displacement in the \( x \) direction

\[ V(x, t) \] velocity vector of a point in the beam

\[ v(x, t) \] beam lateral displacement in the \( y \) direction

\[ V(x, t) \] dimensionless displacement, \( v(x, t)/L \)

\[ w(x, t) \] beam lateral displacement in the \( z \) direction

\[ W(x, t) \] dimensionless displacement, \( w(x, t)/L \)

\[ X, Y, Z \] principal frame coordinates

\[ x, y, z \] fixed frame coordinates

\[ \alpha \] dimensionless rotational speed, \( \Omega L^2 \sqrt{\rho(0)A(0)/[E(0)I_{yy}(0)]} \)

\[ \beta_1, \beta_2, \beta_3, \beta_4 \] dimensionless rotational and translational spring constants at the root of beam in the \( y \) and \( z \) directions, respectively,

\[ K_{y0}L/[E(0)I_{yy}(0)], K_{zT}L/[E(0)I_{yy}(0)], K_{y0}L/[E(0)I_{yy}(0)], \]

\[ K_{yT}L/[E(0)I_{yy}(0)] \]

\[ \gamma_{11} \] dimensionless parameter, \( \beta_i/(1 + \beta_i) \)

\[ \gamma_{12} \] dimensionless parameter, \( 1/(1 + \beta_i) \)
\( \delta_1^*, \delta_2^* \) dimensionless rotatory inertia of the tip mass about the \( Y \) and \( Z \) axis, respectively, \( J_{YY^*}/[\rho(0)A(0)L^3], J_{ZZ^*}/[\rho(0)A(0)L^3] \)
\( \delta_1, \delta_2 \) dimensionless rotatory inertia of the tip mass about the \( y \) and \( z \) axis, respectively, \( J_{yy^*}/[\rho(0)A(0)L^3], J_{zz^*}/[\rho(0)A(0)L^3] \),
\( \delta_3 \) dimensionless inertia of the tip mass, \( M^*/[\rho(0)A(0)L] \)
\( \varepsilon_\xi \) normal strain in the \( x \) direction
\( \eta \) the square of the ratio of the width of the cross-section at the root of the beam in the \( y \) direction and \( z \) direction
\( \theta \) setting angle between principal and fixed frames
\( \lambda_1, \lambda_2 \) variable rate of the width of the cross-section of the beam in the \( Y \) and \( Z \) directions, respectively
\( \xi \) dimensionless distance to the root of the beam, \( \xi = x/L \)
\( A \) dimensionless natural frequency, \( \omega L^2 \sqrt{\rho(0)A(0)/[E(0)I_{yy}(0)]} \)
\( \rho(x) \) mass per unit volume of the beam
\( \tau \) dimensionless time, \( t/L^2 \sqrt{E(0)I_{yy}(0)/\rho(0)A(0)} \)
\( \sigma_x \) normal stress in the \( x \) direction
\( \Phi \) tip pretwist angle of the beam, \( \theta(L) \)
\( \varphi \) pretwist angle of the beam
\( \Omega \) angular velocity
\( \omega \) natural frequency
\( \omega \) dimensionless excitation frequency


pretwisted Bernoulli–Euler beam. Young and Lin [21] used the Galerkin method to study the stability of a cantilever tapered pretwisted beam with varying speed. Yoo and Kwak and Chung [22] used the Rayleigh–Ritz method to study the vibration of a rotating pretwisted blade with a concentrated mass. Lin [23] derived the frequency equation of the system and expressed in terms of the transition matrix of the vector governing equation. The influence of the rotatory inertia and the phenomenon of divergence instability had been investigated. Further, Lee et al. [24] investigated the free vibration of a rotating nonuniform beam with a tip mass, arbitrary pretwist and an elastically restrained root by using the method given by Lin [23]. There still is no study on the forced vibration and static response of rotating nonuniformly pretwisted beam with an elastically restrained root and a tip mass. The corresponding Green function of the system has not been discovered.

In this paper, the governing differential equations for the coupled bending–bending-extensional vibration of a rotating nonuniform beam with a tip mass, arbitrary pretwist and an elastically restrained root, subjected to the external transverse forces in the $y$ and $z$ directions, are derived by using Hamilton’s principle. For an inextensional beam, without considering the Coriolis force effect, three coupled bending–bending-extensional governing differential equations are reduced to two coupled bending–bending equations and the centrifugal force is obtained. Using the mode expansion method derives the closed-form solutions of the dynamic and static systems. The orthogonal condition for the eigenfunctions of the system is discovered. The self-adjointness of the system is proved. Moreover, the Green functions of the system are derived. The symmetric properties of the Green functions are investigated. Finally, the frequencies and eigenfunctions are determined by using the method given by Lee et al. [24] and the numerical frequency response on the steady response of the beam is determined.

2. Dynamic analysis

2.1. Governing equations and boundary conditions

Consider the dynamic response of a pretwisted and doubly symmetric nonuniform beam with tip mass, subjected to any transverse external forces, elastically restrained, and mounted on a hub that rotates with constant angular velocity, as shown in Fig. 1. The displacement fields of the beam are

$$u = u_0(x,t) - z \frac{\partial w}{\partial x} - y \frac{\partial v}{\partial x}, \quad v = v(x,t), \quad w = w(x,t)$$

(1)

The velocity vector of a point $(x,y,z)$ in the beam is given by

$$\vec{V} = \left[ \frac{\partial u}{\partial t} + \Omega \sin \theta (z + w) + \Omega \cos \theta (y + v) \right] \hat{i} + \left[ \frac{\partial v}{\partial t} - \Omega \cos \theta (x + y) \right] \hat{j} + \left[ \frac{\partial w}{\partial t} - \Omega \sin \theta (x + R + u) \right] \hat{k}$$

(2)
The potential energy $\tilde{U}$ and the kinetic energy $\tilde{T}$ of the beam are

$$\tilde{U} = \frac{1}{2} \int_0^L \int_A \sigma_{ex} \, dx \, dA + K_{z\theta} \left[ \frac{\partial w(0,t)}{\partial x} \right]^2 + \frac{1}{2} K_{zT} w^2(0,t)$$

$$+ \frac{1}{2} K_{y\theta} \left[ \frac{\partial v(0,t)}{\partial x} \right]^2 + \frac{1}{2} K_{yT} v^2(0,t) - \int_0^L \left[ p(x,t)v(x,t) + q(x,t)w(x,t) \right] \, dx$$

and

$$\tilde{T} = \frac{1}{2} \int_0^L \int_A \rho \vec{V} \cdot \vec{V} \, dx \, dA + \frac{1}{2} M^* [\vec{V} \cdot \vec{V}]_{x=L,y=0,z=0}$$

$$+ \frac{1}{2} J_{yy}^* \left[ \Omega \sin \theta - \frac{\partial^2 w(L,t)}{\partial x \partial t} \right]^2 + \frac{1}{2} J_{zz}^* \left[ -\Omega \cos \theta - \frac{\partial^2 v(L,t)}{\partial x \partial t} \right]^2$$

Application of Hamilton’s principle yields three coupled governing differential equations and ten elastic boundary conditions. For an inextensional beam, without considering the Coriolis force effect, the governing differential equations and boundary conditions are reduced to two coupled bending-bending equations and eight elastic boundary conditions. Moreover, the centrifugal force is obtained.
The two-coupled dimensionless governing differential equations of the system are obtained

\[
\frac{\partial^2}{\partial \xi^2} \left( B_{yy} \frac{\partial^2 W}{\partial \xi^2} + B_{yz} \frac{\partial^2 V}{\partial \xi^2} \right) - \frac{\partial}{\partial \xi} \left( \vec{n} \frac{\partial W}{\partial \xi} \right) + m \frac{\partial^2 W}{\partial \tau^2} - mx^2 \sin^2 \theta W - mx^2 \sin \theta \cos \theta V = P(\xi, \tau),
\]

\hspace{1cm} (5)

\[
\frac{\partial^2}{\partial \xi^2} \left( B_{yz} \frac{\partial^2 W}{\partial \xi^2} + B_{zz} \frac{\partial^2 V}{\partial \xi^2} \right) - \frac{\partial}{\partial \xi} \left( \vec{n} \frac{\partial V}{\partial \xi} \right) + m \frac{\partial^2 V}{\partial \tau^2} - mx^2 \cos^2 \theta V - mx^2 \sin \theta \cos \theta W = Q(\xi, \tau), \quad \xi \in (0, 1)
\]

\hspace{1cm} (6)

and the associated dimensionless elastic boundary conditions are

At \( \xi = 0 \):

\[
\gamma_{12} \left( B_{yy} \frac{\partial^2 W}{\partial \xi^2} + B_{yz} \frac{\partial^2 V}{\partial \xi^2} \right) - \gamma_{11} \frac{\partial W}{\partial \xi} = 0,
\]

\hspace{1cm} (7)

\[
\gamma_{22} \left[ \frac{\partial}{\partial \xi} \left( B_{yy} \frac{\partial^2 W}{\partial \xi^2} + B_{yz} \frac{\partial^2 V}{\partial \xi^2} \right) - \vec{n} \frac{\partial W}{\partial \xi} \right] + \gamma_{21} W = 0,
\]

\hspace{1cm} (8)

\[
\gamma_{32} \left( B_{yz} \frac{\partial^2 W}{\partial \xi^2} + B_{zz} \frac{\partial^2 V}{\partial \xi^2} \right) - \gamma_{31} \frac{\partial V}{\partial \xi} = 0,
\]

\hspace{1cm} (9)

\[
\gamma_{42} \left[ \frac{\partial}{\partial \xi} \left( B_{yz} \frac{\partial^2 W}{\partial \xi^2} + B_{zz} \frac{\partial^2 V}{\partial \xi^2} \right) - \vec{n} \frac{\partial V}{\partial \xi} \right] + \gamma_{41} W = 0.
\]

\hspace{1cm} (10)

At \( \xi = 1 \):

\[
B_{yy} \frac{\partial^2 W}{\partial \xi^2} + B_{yz} \frac{\partial^2 V}{\partial \xi^2} + \delta_1 \frac{\partial^3 W}{\partial \xi \partial \tau^2} = 0,
\]

\hspace{1cm} (11)

\[
\frac{\partial}{\partial \xi} \left( B_{yy} \frac{\partial^2 W}{\partial \xi^2} + B_{yz} \frac{\partial^2 V}{\partial \xi^2} \right) - \vec{n} \frac{\partial W}{\partial \xi} + \delta_3 \left( x^2 \sin^2 \theta W + x^2 \sin \theta \cos \theta V - \frac{\partial^2 W}{\partial \tau^2} \right) = 0,
\]

\hspace{1cm} (12)

\[
B_{yz} \frac{\partial^2 W}{\partial \xi^2} + B_{zz} \frac{\partial^2 V}{\partial \xi^2} + \delta_2 \frac{\partial^3 V}{\partial \xi \partial \tau^2} = 0,
\]

\hspace{1cm} (13)

\[
\frac{\partial}{\partial \xi} \left( B_{yz} \frac{\partial^2 W}{\partial \xi^2} + B_{zz} \frac{\partial^2 V}{\partial \xi^2} \right) - \vec{n} \frac{\partial V}{\partial \xi} + \delta_3 \left( x^2 \sin \theta \cos \theta W + x^2 \cos^2 \theta V - \frac{\partial^2 V}{\partial \tau^2} \right) = 0.
\]

\hspace{1cm} (14)
The dimensionless initial conditions of the motion are assumed as follows,

\[ W(\xi, \tau) = W_0(\xi), \quad V(\xi, \tau) = V_0(\xi), \]

\[ \frac{\partial W(\xi, \tau)}{\partial \tau} = \dot{W}_0(\xi), \quad \frac{\partial V(\xi, \tau)}{\partial \tau} = \dot{V}_0(\xi). \]  

(15)

2.2. Solution method

2.2.1. Orthogonal condition

In this paper, the solutions for Eqs. (5)–(15), \( W(\xi, \tau) \) and \( V(\xi, \tau) \) can be obtained by using the method of eigenfunction expansion. Assume that dimensionless natural frequency \( \Lambda_n \) is the \( n \)th eigenvalue and \([\hat{W}_n(\xi) \quad \hat{V}_n(\xi)]^T\) be the \( n \)th eigenfunction of the system, where the superscript \( T \) is the transpose of a matrix. The governing characteristic differential equation can be expressed as

\[ \{[\hat{L}] - \Lambda_n^2[\hat{M}]\} \begin{bmatrix} \hat{W}_n \\ \hat{V}_n \end{bmatrix} = 0, \]  

(16)

where the differential operators \([\hat{L}]\) and \([\hat{M}]\) are

\[ [\hat{L}] = \begin{bmatrix} \hat{L}_{11} & \hat{L}_{12} \\ \hat{L}_{21} & \hat{L}_{22} \end{bmatrix}, \]

\[ \hat{L}_{11} = \frac{d^2}{d\xi^2} \left( B_{yy} \frac{d^2}{d\xi^2} \right) - \frac{d}{d\xi} \left( \hat{n} \frac{d}{d\xi} \right) - mx^2 \sin^2 \theta, \]

\[ \hat{L}_{12} = \frac{d^2}{d\xi^2} \left( B_{yz} \frac{d^2}{d\xi^2} \right) - mx^2 \sin \theta \cos \theta, \]

\[ \hat{L}_{21} = \frac{d^2}{d\xi^2} \left( B_{yz} \frac{d^2}{d\xi^2} \right) - mx^2 \sin \theta \cos \theta, \]

\[ \hat{L}_{22} = \frac{d^2}{d\xi^2} \left( B_{zz} \frac{d^2}{d\xi^2} \right) - \frac{d}{d\xi} \left( \hat{n} \frac{d}{d\xi} \right) - mx^2 \cos^2 \theta, \]  

(17)

and

\[ [\hat{M}] = \begin{bmatrix} m(\zeta) & 0 \\ 0 & m(\zeta) \end{bmatrix}. \]  

(18)
Taking the inner product of the eigenfunctions $[\hat{W}_n(\zeta) \hat{V}_n(\zeta)]^T$ and the differential operators $[\hat{L}], [\hat{M}]$, one can show that

$$\int_0^1 [\hat{W}_j(\zeta) \hat{V}_j(\zeta)][\hat{L}] \left[ \begin{array}{c} \hat{W}_n(\zeta) \\ \hat{V}_n(\zeta) \end{array} \right] d\zeta = \int_0^1 [\hat{W}_n(\zeta) \hat{V}_n(\zeta)][\hat{M}] \left[ \begin{array}{c} \hat{W}_j(\zeta) \\ \hat{V}_j(\zeta) \end{array} \right] d\zeta, \quad (19)$$

and

$$\int_0^1 [\hat{W}_j(\zeta) \hat{V}_j(\zeta)][\hat{L}] \left[ \begin{array}{c} \hat{W}_n(\zeta) \\ \hat{V}_n(\zeta) \end{array} \right] d\zeta = \int_0^1 [\hat{W}_n(\zeta) \hat{V}_n(\zeta)][\hat{L}] \left[ \begin{array}{c} \hat{W}_j(\zeta) \\ \hat{V}_j(\zeta) \end{array} \right] d\zeta + \hat{A}, \quad (20)$$

where

$$\hat{A} = \hat{W}_j \left[ \frac{d}{d\zeta} \left( B_{yy} \frac{d^2 \hat{W}_n}{d\zeta^2} \right) + \frac{d}{d\zeta} \left( B_{yz} \frac{d^2 \hat{V}_n}{d\zeta^2} \right) - \bar{n} \frac{d\hat{W}_n}{d\zeta} \right] \bigg|_0^1$$

$$+ \hat{V}_j \left[ \frac{d}{d\zeta} \left( B_{yz} \frac{d^2 \hat{W}_n}{d\zeta^2} \right) + \frac{d}{d\zeta} \left( B_{zz} \frac{d^2 \hat{V}_n}{d\zeta^2} \right) - \bar{n} \frac{d\hat{V}_n}{d\zeta} \right] \bigg|_0^1$$

$$- \frac{d\hat{W}_j}{d\zeta} \left[ \frac{d}{d\zeta} \left( B_{yy} \frac{d^2 \hat{W}_n}{d\zeta^2} \right) + \frac{d}{d\zeta} \left( B_{yz} \frac{d^2 \hat{V}_n}{d\zeta^2} \right) \right] \bigg|_0^1$$

$$- \frac{d\hat{V}_j}{d\zeta} \left[ \frac{d}{d\zeta} \left( B_{yz} \frac{d^2 \hat{W}_n}{d\zeta^2} \right) + \frac{d}{d\zeta} \left( B_{zz} \frac{d^2 \hat{V}_n}{d\zeta^2} \right) \right] \bigg|_0^1$$

$$+ \frac{d\hat{W}_n}{d\zeta} \left[ \frac{d}{d\zeta} \left( B_{yy} \frac{d^2 \hat{W}_j}{d\zeta^2} \right) + \frac{d}{d\zeta} \left( B_{yz} \frac{d^2 \hat{V}_j}{d\zeta^2} \right) \right] \bigg|_0^1$$

$$+ \frac{d\hat{V}_n}{d\zeta} \left[ \frac{d}{d\zeta} \left( B_{yz} \frac{d^2 \hat{W}_j}{d\zeta^2} \right) + \frac{d}{d\zeta} \left( B_{zz} \frac{d^2 \hat{V}_j}{d\zeta^2} \right) \right] \bigg|_0^1$$

$$- \hat{W}_n \left[ \frac{d}{d\zeta} \left( B_{yy} \frac{d^2 \hat{W}_j}{d\zeta^2} \right) + \frac{d}{d\zeta} \left( B_{yz} \frac{d^2 \hat{V}_j}{d\zeta^2} \right) - \bar{n} \frac{d\hat{W}_j}{d\zeta} \right] \bigg|_0^1$$

$$- \hat{V}_n \left[ \frac{d}{d\zeta} \left( B_{yz} \frac{d^2 \hat{W}_j}{d\zeta^2} \right) + \frac{d}{d\zeta} \left( B_{zz} \frac{d^2 \hat{V}_j}{d\zeta^2} \right) - \bar{n} \frac{d\hat{V}_j}{d\zeta} \right] \bigg|_0^1 \quad (21)$$
and $\hat{A}$ vanishes because of the boundary conditions (7)–(14). Thus, the self-adjointness of the system is proved. The following orthogonality condition is obtained:

$$\int_0^1 M(\xi)[\hat{W}_j(\xi)\hat{W}_n(\xi) + \hat{V}_j(\xi)\hat{V}_n(\xi)]\,d\xi = \begin{cases} 0, & j \neq n \\ \varepsilon_n, & j = n \end{cases}$$  \hfill (22)

where $\varepsilon_n$ is a real number.

### 2.2.2. General solution

The solutions $W(\xi, \tau)$ and $V(\xi, \tau)$ for (5)–(15) can be expressed in the following eigenfunction expansion form:

$$\begin{bmatrix} W(\xi, \tau) \\
V(\xi, \tau) \end{bmatrix} = \sum_{n=1}^{\infty} T_n(\tau) \begin{bmatrix} \hat{W}_n(\xi) \\
\hat{V}_n(\xi) \end{bmatrix},$$  \hfill (23)

Substituting it back to the governing equations (5) and (6) and the initial conditions (15), then multiplying by $[\hat{W}_n(\xi) \hat{V}_n(\xi)]$, and integrating in accordance with the orthogonal condition (22), one obtains

$$\frac{d^2 T_n}{d\tau^2} + A_n^2 T_n = P_n^*(\tau),$$  \hfill (24)

where

$$P_n^*(\tau) = \frac{1}{\varepsilon_n} \int_0^1 [\hat{W}_n(\xi)P(\xi, \tau) + \hat{V}_n(\xi)Q(\xi, \tau)]\,d\xi,$$  \hfill (25)

The corresponding initial conditions are

$$T_n(0) = \frac{1}{\varepsilon_n} \int_0^1 m(\xi)[\hat{W}_n(\xi)W(\xi, 0) + \hat{V}_n(\xi)V(\xi, 0)]\,d\xi,$$  \hfill (26)

$$\frac{dT_n(0)}{d\tau} = \frac{1}{\varepsilon_n} \int_0^1 m(\xi) \left[ \hat{W}_n(\xi) \frac{\partial W(\xi, 0)}{\partial \tau} + \hat{V}_n(\xi) \frac{\partial V(\xi, 0)}{\partial \tau} \right]\,d\xi,$$  \hfill (27)

The solution is

$$T_n(\tau) = T_n(0)\cos A_n\tau + \frac{1}{A_n} \frac{dT_n(0)}{d\tau} \sin A_n\tau + \frac{1}{A_n} \int_0^\tau P_n^*(\zeta) \sin A_n(\tau - \zeta)\,d\zeta.$$  \hfill (28)

After substituting the solution (28) back into the general solution (23), the general forced response of the system with elastic boundary conditions is obtained.
3. Static analysis

3.1. Governing equations and boundary conditions

Consider the static behavior of rotating pretwisted beams. The corresponding dimensionless governing differential equations of the system are reduced from Eqs. (5) and (6) to be

\[
\frac{d^2}{d\zeta^2} \left( B_{yy} \frac{d^2 W}{d\zeta^2} + B_{yz} \frac{d^2 V}{d\zeta^2} \right) - \frac{d}{d\zeta} \left( \tilde{n} \frac{dW}{d\zeta} \right) - mx^2 \sin^2 \theta W - mx^2 \sin \theta \cos \theta V = P(\zeta),
\]

\[
\frac{d^2}{d\zeta^2} \left( B_{yz} \frac{d^2 W}{d\zeta^2} + B_{zz} \frac{d^2 V}{d\zeta^2} \right) - \frac{d}{d\zeta} \left( \tilde{n} \frac{dV}{d\zeta} \right) - mx^2 \cos^2 \theta V - mx^2 \sin \theta \cos \theta W = Q(\zeta), \quad \zeta \in (0, 1)
\]

and the associated dimensionless elastic boundary conditions are obtained from Eqs. (7)–(13).

At \( \zeta = 0 \):

\[
\gamma_{12} \left( B_{yy} \frac{d^2 W}{d\zeta^2} + B_{yz} \frac{d^2 V}{d\zeta^2} \right) - \gamma_{11} \frac{dW}{d\zeta} = 0,
\]

\[
\gamma_{22} \left[ \frac{d}{d\zeta} \left( B_{yy} \frac{d^2 W}{d\zeta^2} + B_{yz} \frac{d^2 V}{d\zeta^2} \right) - \tilde{n} \frac{dW}{d\zeta} \right] + \gamma_{21} W = 0,
\]

\[
\gamma_{32} \left( B_{yz} \frac{d^2 W}{d\zeta^2} + B_{zz} \frac{d^2 V}{d\zeta^2} \right) - \gamma_{31} \frac{dV}{d\zeta} = 0,
\]

\[
\gamma_{42} \left[ \frac{d}{d\zeta} \left( B_{yz} \frac{d^2 W}{d\zeta^2} + B_{zz} \frac{d^2 V}{d\zeta^2} \right) - \tilde{n} \frac{dV}{d\zeta} \right] + \gamma_{41} W = 0.
\]

At \( \zeta = 1 \):

\[
B_{yy} \frac{d^2 W}{d\zeta^2} + B_{yz} \frac{d^2 V}{d\zeta^2} = 0,
\]

\[
\frac{d}{d\zeta} \left( B_{yy} \frac{d^2 W}{d\zeta^2} + B_{yz} \frac{d^2 V}{d\zeta^2} \right) - \tilde{n} \frac{dW}{d\zeta} + \delta_3 (x^2 \sin^2 \theta W + x^2 \sin \theta \cos \theta V) = 0,
\]

\[
B_{yz} \frac{d^2 W}{d\zeta^2} + B_{zz} \frac{d^2 V}{d\zeta^2} = 0,
\]

\[
\frac{d}{d\zeta} \left( B_{yz} \frac{d^2 W}{d\zeta^2} + B_{zz} \frac{d^2 V}{d\zeta^2} \right) - \tilde{n} \frac{dV}{d\zeta} + \delta_3 (x^2 \sin \theta \cos \theta W + x^2 \cos^2 \theta V) = 0.
\]
3.2. Green's functions

The governing differential equations (29) and (30) are written as

\[
[\hat{L}] \begin{bmatrix} W(\xi) \\ V(\xi) \end{bmatrix} = \begin{bmatrix} P(\xi) \\ Q(\xi) \end{bmatrix},
\]

(39)

where matrix differential operator \([\hat{L}]\) is the same as Eq. (17). The self-adjointness of the system is proved before. Then the following relation can be obtained

\[
\int_{0}^{1} [W_a(\xi) \ V_a(\xi)] [\hat{L}] \begin{bmatrix} W_b(\xi) \\ V_b(\xi) \end{bmatrix} \mathrm{d}\xi = \int_{0}^{1} [W_b(\xi) \ V_b(\xi)] [\hat{L}] \begin{bmatrix} W_a(\xi) \\ V_a(\xi) \end{bmatrix} \mathrm{d}\xi.
\]

(40)

Let

\[
[\hat{L}] \begin{bmatrix} G_{11}(\xi, \chi) \\ G_{12}(\xi, \chi) \end{bmatrix} = \begin{bmatrix} \delta(\xi - \chi) \\ 0 \end{bmatrix},
\]

and

\[
[\hat{L}] \begin{bmatrix} G_{21}(\xi, \zeta) \\ G_{22}(\xi, \zeta) \end{bmatrix} = \begin{bmatrix} 0 \\ \delta(\xi - \zeta) \end{bmatrix},
\]

(41)

where \(G_{ij}\) are Green’s functions. Then the general static solution can be obtained

\[
\begin{bmatrix} W(\xi) \\ V(\xi) \end{bmatrix} = \int_{0}^{1} P(\zeta) \begin{bmatrix} G_{11}(\xi, \zeta) \\ G_{12}(\xi, \zeta) \end{bmatrix} + Q(\zeta) \begin{bmatrix} G_{21}(\xi, \zeta) \\ G_{22}(\xi, \zeta) \end{bmatrix} \mathrm{d}\zeta,
\]

(42)

Letting \([W_a(\xi) \ V_a(\xi)]^T = [W_b(\xi) \ V_b(\xi)]^T = [G_{11}(\xi, \chi) \ G_{12}(\xi, \chi)]^T\) and substituting these into Eq. (40), the following symmetric property is obtained

\[
G_{11}(\xi, \chi) = G_{11}(\chi, \xi).
\]

(43)

Similarly, letting \([W_a(\xi) \ V_a(\xi)]^T = [W_b(\xi) \ V_b(\xi)]^T = [G_{21}(\xi, \chi) \ G_{22}(\xi, \chi)]^T\) and substituting these into Eq. (40), the following symmetric property is obtained

\[
G_{22}(\xi, \chi) = G_{22}(\chi, \xi).
\]

(44)

Finally, letting \([W_a(\xi) \ V_a(\xi)]^T\) and \([W_b(\xi) \ V_b(\xi)]^T\) are \([G_{11}(\xi, \chi) \ G_{12}(\xi, \chi)]^T\) and \([G_{21}(\xi, \zeta) \ G_{22}(\zeta, \zeta)]^T\), respectively, and substituting these into Eq. (40), the following symmetric property is obtained

\[
G_{21}(\zeta, \chi) = G_{12}(\chi, \zeta).
\]

(45)
3.3. General solution

The general solution \([W(\xi) \ V(\xi)]^T\) can be expressed in the following eigenfunction expansion form:

\[
\begin{bmatrix} W(\xi) \\ V(\xi) \end{bmatrix} = \sum_{n=1}^{\infty} \lambda_n \begin{bmatrix} \hat{W}_n(\xi) \\ \hat{V}_n(\xi) \end{bmatrix},
\]

where \([\hat{W}_n(\xi) \ \hat{V}_n(\xi)]^T\) is the eigenfunction. Substituting it back to the governing equations (29) and (30), then multiplying by \([\hat{W}_n(\xi) \ \hat{V}_n(\xi)]\), and integrating in accordance with the orthogonal condition (22), one obtains

\[
\lambda_n = \frac{1}{A_n^{D_s} S_n^k} \int_0^1 \left[ \hat{W}_n(\xi)P(\xi) + \hat{V}_n(\xi)Q(\xi) \right] d\xi,
\]

(47)

After substituting Eq. (47) back into Eq. (46), the general static solution (42) is obtained and the associated Green’s functions are

\[
\begin{bmatrix} G_{11}(\xi, \zeta) \\ G_{12}(\xi, \zeta) \end{bmatrix} = \sum_{n=1}^{\infty} \frac{\hat{W}_n(\zeta)}{A_n^{D_s} S_n^k} \begin{bmatrix} \hat{W}_n(\xi) \\ \hat{V}_n(\xi) \end{bmatrix},
\]

\[
\begin{bmatrix} G_{21}(\xi, \zeta) \\ G_{22}(\xi, \zeta) \end{bmatrix} = \sum_{n=1}^{\infty} \frac{\hat{V}_n(\zeta)}{A_n^{D_s} S_n^k} \begin{bmatrix} \hat{W}_n(\xi) \\ \hat{V}_n(\xi) \end{bmatrix}.
\]

(48)

It should be noted that the symmetric properties (43)–(45) of Green’s functions are satisfied.

4. Numerical results and discussion

Consider rotating pretwisted doubly tapered beam. The following parameters are given:

\[
B_{\partial y}(\xi) = \eta(1 + \lambda_1 \xi^3)(1 + \lambda_2 \xi) \sin^2 \varphi + (1 + \lambda_1 \xi)(1 + \lambda_2 \xi)^3 \cos^2 \varphi,
\]

\[
B_{\partial z}(\xi) = \eta(1 + \lambda_1 \xi^3)(1 + \lambda_2 \xi) \cos^2 \varphi + (1 + \lambda_1 \xi)(1 + \lambda_2 \xi)^3 \sin^2 \varphi,
\]

\[
B_{\partial \zeta}(\xi) = 0.5[\eta(1 + \lambda_1 \xi^3)(1 + \lambda_2 \xi) - (1 + \lambda_1 \xi)(1 + \lambda_2 \xi)^3] \sin^2(2\varphi),
\]

\[
\delta_2^* = k_2 \delta_3,
\]

\[
\delta_1^* = k_1 \delta_2^*,
\]

\[
\delta_1 = \delta_2^* \sin^2 \Phi + \delta_1^* \cos^2 \Phi,
\]

\[
\delta_2 = \delta_2^* \cos^2 \Phi + \delta_1^* \sin^2 \Phi.
\]

(49)

It is also assumed that the beam is subjected to two external harmonic transverse concentrated forces

\[
P(\xi, \tau) = c_p \sin \sigma \tau \delta(\xi - \xi_0), \quad Q(\xi, \tau) = c_q \sin \sigma \tau \delta(\xi - \xi_0),
\]

(50)
Substituting Eq. (50) into Eqs. (24) and (25), one obtains

\[ P_n'(\tau) = \frac{1}{e_n} \left[ c_p \hat{W}_n(\xi_0) + c_q \hat{V}_n(\xi_0) \right] \sin \omega \tau = \tilde{p}_n \sin \omega \tau \] (51)

\[ T_n(\tau) = \frac{\tilde{p}_n}{A_n^2 - \omega^2} \sin \omega \tau. \] (52)

The frequencies \( A_n \) and mode shapes \( [\hat{W}_n \hat{V}_n]^T \) of the system can be determined by using the semi-analytical method by Lee et al. [24]. Fig. 2 shows the influence of the excitation frequency \( \omega \) of the transverse forces on the steady response of the tip of a uniform beam clamped on the hub, pretwisted uniformly, with various geometric parameter of the cross-section \( \eta \). It can be observed from Eq. (52) and in Fig. 2 that when the excitation frequency \( \omega \) approaches the natural frequencies, the response increases rapidly and becomes infinite as \( \omega \) coincide with the natural frequencies. Moreover, It is shown in Fig. 2 that the natural frequencies increase as \( \eta \) is increased.

5. Conclusion

In this paper, the governing differential equations for the coupled bending–bending vibration of a rotating nonuniform beam with a tip mass, arbitrary pretwist and an elastically restrained root, subjected to the external transverse forces in the \( y \) and \( z \) directions, and rotating at a constant angular velocity, are derived by using Hamilton’s principle. The closed-form solutions of the dynamic and static systems are proposed by using the mode expansion method. The orthogonal condition for the eigenfunctions of the system is discovered. The self-adjointness of the system is proved. Moreover,
the matrix Green functions of the static system are obtained. The symmetric properties of the Matrix Green functions are revealed.

Acknowledgements

The support of the National Science Council of Taiwan, ROC, is gratefully acknowledged (Grant number: Nsc91-2212-E168-005)

References


