Exact solutions for extensible circular curved Timoshenko beams with nonhomogeneous elastic boundary conditions

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Summary. A generalized Green function of nth-order ordinary differential equation with forcing function composed of the delta function and its derivatives is obtained. The generalized Green function can be easily and effectively applied to both the boundary value problems and the initial value problems. The generalized Green function is expressed in terms of n linearly independent normalized homogeneous solutions. It is the generalization of those given by Pan and Hohenstein, and Kanwal. Accordingly, the exact solution for static analysis of an extensible circular curved Timoshenko beam with general nonhomogeneous elastic boundary conditions, subjected to any transverse, tangential and moment loads is obtained. The three coupled governing differential equations are uncoupled into one complete sixth-order ordinary differential characteristic equation in the tangential displacement. The explicit relations between the angle of rotation due to bending, the transverse displacement and the tangential displacement are obtained. The deflection curves due to a unit generalized displacement at nodal coordinate, and the exact element stiffness matrix are derived based on the solution for the general system. A finite element method can be developed based on the results for the dynamic analysis. Meanwhile, the stiffness locking phenomena accompanied in some other curved beam element methods does not exist in the proposed method.

1 Introduction

Many physical problems such as diffusion, heat conduction, vibration, wave propagation etc., can be described by a nth-order linear ordinary differential equation. Pan and Hohenstein [1] offered a method of solution for a fourth-order ordinary differential equation with the forcing terms which are composed of the delta function and its derivatives. Kanwal [2] and Yang and Tan [3] found the matrix Green function of a matrix differential equation without the derivatives of delta function. It is obviously difficult to calculate the fundamental matrix and its inverse whose dimensions are large. Stakgold [4] discussed just some properties of Green’s function of an ordinary differential equation. So far, no generalized Green function of a nth-order, linear ordinary differential equation with forcing terms which are composed of the delta function and its derivatives, has been obtained.

Curved beams, which have importance in many practical applications such as aircraft structures, bridges, was a textbook subject for a long time. An interesting review of the subject can be found in the reviewer papers [5], [6]. The exact solutions for static analysis of extensional circular curved Timoshenko beams with some special conditions has been obtained by using the Castigliano’s theorem [7]. Fetahlioglu and Meyers [8] found the analytical solution for the displacement and stress results for specific Bernoulli-Euler beams. Gauthier and Jahs-

The purpose of this paper is to find the generalized Green function of a nth-order ordinary differential equation with forcing function composed of the delta function and its derivatives. The generalized Green function can be easily and effectively applied to both the boundary value problems and the initial value problems. The Green function is expressed in terms of the n linearly independent normalized homogeneous solutions. The method of solution presented here is more simple than those given by Pan and Hohenstein [1] and Yang and Tan [3]. It is the generalization of that given by Pan and Hohenstein [1] and Kanwal [2]. Accordingly, a systematic theoretical development of the static analysis of extensional circular curved Timoshenko beams with general nonhomogeneous elastic boundary conditions is presented. The exact solution for the analysis is obtained. The three coupled governing differential equations are uncoupled into one complete sixth-order ordinary differential characteristic equation in the tangential displacement. The implicit relations between the transverse displacement, the tangential displacement, and the angle of rotation due to bending are established. The deflection curves due to a unit generalized displacement at nodal coordinate, and the exact element stiffness matrix are derived based on the solution for the general system. Moreover, the consistent mass matrix can be constructed via the deflection curves. A finite element method can be developed based on the results for the dynamic analysis. Meanwhile, the stiffness locking phenomena accompanied in some other curved beam element methods does not exist in the proposed method.

2 General solution

Consider a nth-order ordinary differential equation

$$q_n \frac{d^n w}{d \xi^n} + q_{n-1} \frac{d^{n-1} w}{d \xi^{n-1}} + \ldots + q_1 \frac{dw}{d \xi} + q_0 w = \sum_{i=0}^{n-1} \frac{d^i p_i(\xi)}{d \xi^i}, \quad \xi \in (0, 1) \quad (2.1)$$

where the coefficients \( \{ q_i \} \) are constants on the closure domain [0, 1], and the leading coefficient \( q_n \) does not vanish anywhere on closure domain.

The general solution \( w(\xi) \) of Eq. (2.1) can be written as

$$w(\xi) = w_p(\xi) + \sum_{i=1}^{n} C_i V_i(\xi), \quad (2.2)$$

where \( w_p(\xi) \) is the particular solution. \( \{ C_i \} \) are the constants to be determined. \( \{ V_i(\xi) \} \) are n linearly independent homogeneous solutions of Eq. (2.1), which satisfy the following norma-
lized condition

\[
\begin{bmatrix}
V_1(0) & V_2(0) & \cdots & V_{n-1}(0) & V_n(0) \\
V_1'(0) & V_2'(0) & \cdots & V_{n-1}'(0) & V_n'(0) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
V_1^{(n-2)}(0) & V_2^{(n-2)}(0) & \cdots & V_{n-1}^{(n-2)}(0) & V_n^{(n-2)}(0) \\
V_1^{(n-1)}(0) & V_2^{(n-1)}(0) & \cdots & V_{n-1}^{(n-1)}(0) & V_n^{(n-1)}(0)
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \\
0 & 0 & \cdots & 0 & 1
\end{bmatrix},
\tag{2.3}
\]

The particular solution \(w_p(\xi)\) of Eq. (2.1) can be written as

\[
w_p(\xi) = \sum_{i=0}^{n-1} \int p_i(x) E_i(\xi - x) \, dx,
\tag{2.4}
\]

where \(\{E_i(\xi)\}\) are the Green functions of the following equations, respectively

\[
q_n \frac{d^n}{d\xi^n} E_i + q_{n-1} \frac{d^{n-1}}{d\xi^{n-1}} E_i + \cdots + q_1 \frac{d}{d\xi} E_i + q_0 E_i = \frac{d^\delta}{d\xi^i}, \quad i = 0, 1, 2, \ldots, n - 1
\tag{2.5}
\]

in which \(\delta\) is the Dirac delta function. The Green functions \(\{E_i(\xi)\}\) can be written as

\[
E_0(\xi) = \tilde{E}_0(\xi),
\]

\[
E_i(\xi) = \tilde{E}_i(\xi) - \sum_{j=1}^{i} \frac{q_{n-j}}{q_n} E_{i-j}(\xi), \quad i = 1, 2, \ldots, n - 1,
\tag{2.6}
\]

where \(\{\tilde{E}_i(\xi)\}\) are the Green functions of the following equations, respectively

\[
q_n \frac{d^n}{d\xi^n} E_i + q_{n-1} \frac{d^{n-1}}{d\xi^{n-1}} E_i + \cdots + q_1 \frac{d}{d\xi} E_i + q_0 E_i = \frac{d^\delta}{d\xi^i} + \sum_{j=0}^{n-i-1} \frac{q_{n-i-j}}{q_n} \frac{d^\delta}{d\xi^j}, \quad i = 0, 1, 2, \ldots, n - 1
\tag{2.7}
\]

The Green functions \(\{\tilde{E}_i(\xi)\}\) can be obtained

\[
\tilde{E}_i(\xi) = G_i(\xi) H(\xi), \quad i = 0, 1, 2, \ldots, n - 1
\tag{2.8}
\]

where \(H(\xi)\) is the Heaviside function, if the function \(\{G_i(\xi)\}\) satisfy the following conditions

\[
\frac{d^j G_i}{d\xi^j} \bigg|_{\xi=0} = 0, \quad j \neq n - i - 1,
\tag{2.9}
\]

\[
\frac{d^{n-i-1} G_i}{d\xi^{n-i-1}} \bigg|_{\xi=0} = \frac{1}{q_n}, \quad i = 0, 1, 2, \ldots, n - 1
\tag{2.10}
\]

in the distributional sense. Letting the function \(G_i(\xi)\) be

\[
G_i(\xi) = \sum_{j=1}^{n} a_{i,j} V_j(\xi),
\tag{2.11}
\]

and substituting Eq. (2.11) into Eqs. (2.8)-(2.9), the coefficients \(\{a_{i,j}\}\) and the Green functions \(\{\tilde{E}_i(\xi)\}\) are obtained respectively

\[
a_{i,n-i} = \frac{1}{q_n},
\]

\[
a_{i,j} = 0, \quad j \neq n - i, \quad i = 0, 1, 2, \ldots, n - 1.
\tag{2.12}
\]

\[
\tilde{E}_i(\xi) = \frac{1}{q_n} V_{n-i}(\xi) H(\xi), \quad i = 0, 1, 2, \ldots, n - 1.
\tag{2.13}
Substituting Eq. (2.13) into Eqs. (2.2), (2.4), and (2.6), the particular solution and the general solution are obtained. Substituting the general solution into the associated boundary conditions, the coefficients \( \{C_i\} \) can be obtained. It should be noted that the generalized Green function can be easily and effectively applied to both the boundary value problems and the initial value problems. Based on the results the analysis of a curved Timoshenko beam is made as following.

3 Exact solutions

3.1 Extensional curved Timoshenko beams

Consider the static response of an extensional circular Timoshenko beam with general non-homogeneous boundary conditions, subjected to any transverse, tangential and moment loads, as shown in Fig. 1. In terms of the following dimensionless quantities,

\[
\begin{align*}
    f_1 &= \frac{F_1}{L}, \\
    f_2 &= \frac{F_2}{L}, \\
    f_3 &= \frac{F_3}{L}, \\
    f_4 &= \frac{F_4}{L}, \\
    f_5 &= \frac{F_5}{L}, \\
    f_6 &= \frac{F_6}{L}, \\
    f_1^* &= \frac{F_1^* L^2}{EI}, \\
    f_2^* &= \frac{F_2^* L^2}{EI}, \\
    f_3^* &= \frac{F_3^* L}{EI}, \\
    f_4^* &= \frac{F_4^* L^2}{EI}, \\
    f_5^* &= \frac{F_5^* L}{EI}, \\
    f_6^* &= \frac{F_6^* L}{EI}, \\
    m(\xi) &= \frac{M(\theta)L^2}{EI}, \\
    p(\xi) &= \frac{P(\theta)L^3}{EI}, \\
    q(\xi) &= \frac{Q(\theta)L^3}{EI}, \\
    u &= \frac{U}{L}, \\
    w &= \frac{W}{L}, \\
    \alpha &= \frac{L}{R}, \\
    \beta_1 &= \frac{K_{UL} L^3}{EI}, \\
    \beta_2 &= \frac{K_{WL} L^3}{EI}, \\
    \beta_3 &= \frac{K_{UL} L^3}{EI}, \\
    \beta_4 &= \frac{K_{WR} L^3}{EI}, \\
    \beta_5 &= \frac{K_{WR} L^3}{EI}, \\
    \beta_6 &= \frac{K_{BR} L^3}{EI}, \\
    \mu &= \frac{EI}{\kappa G A L^3}, \\
    \zeta &= \frac{AL^2}{I}, \\
    \xi &= \frac{\theta}{\alpha},
\end{align*}
\]

where \( W(\theta) \) and \( U(\theta) \) are the tangential and transverse displacements, respectively. \( \Psi(\theta) \) is the angle of rotation due to bending. \( P(\theta), Q(\theta) \) and \( M(\theta) \) are the applied transverse, tangential and moment loads, respectively. \( E, G, \kappa, I \) and \( A \) denote Young’s modulus, shear modulus, shear correction factor, moment of inertia and area per unit length, respectively. \( K_{UL}, K_{WL} \) and \( K_{BL} \) and \( K_{UR}, K_{WR} \) and \( K_{BR} \) are the transverse translational spring constants, the tangential translational spring constants and the rotational spring constants at \( \theta = 0 \) and \( \theta = \alpha \), respectively. \( R \) and \( L \) are the radius and the length of the circular beam, respectively. \( F_1, F_2, F_3, F_1^*, F_2^* \) and \( F_3^* \), and \( F_4, F_5, F_6, F_4^*, F_5^* \) and \( F_6^* \) are the transverse displacements, the tangential displacements, the slopes, the shear forces, the tangential forces and the
moment loads at the left end and the right end of the beam, respectively. The coupled governing
differential equations are

\[
- \frac{1}{\mu} \frac{d}{d\xi} \left( \frac{du}{d\xi} + \alpha w - \psi \right) - \zeta \alpha \left( \frac{dw}{d\xi} - \alpha u \right) = p(\xi), \quad (3.2)
\]

\[
\alpha \left( \frac{du}{d\xi} + \alpha w - \psi \right) - \frac{d}{d\xi} \left( \frac{dw}{d\xi} - \alpha u \right) = q(\xi), \quad (3.3)
\]

\[
- \frac{1}{\mu} \left( \frac{du}{d\xi} + \alpha w - \psi \right) - \frac{d^2\psi}{d\xi^2} = m(\xi). \quad (3.4)
\]

The associated boundary condition are

at \( \xi = 0 \):

\[
\gamma_{12} \frac{1}{\mu} \left( \frac{du}{d\xi} + \alpha w - \psi \right) - \gamma_{11} u = -\gamma_{11} f_1 - \gamma_{12} f_1^*, \quad (3.5)
\]

\[
\gamma_{22} \left( \frac{dw}{d\xi} - \alpha u \right) - \gamma_{21} w = -\gamma_{21} f_2 - \gamma_{22} f_2^*, \quad (3.6)
\]

\[
\gamma_{32} \frac{d\psi}{d\xi} - \gamma_{31} \psi = -\gamma_{31} f_3 - \gamma_{32} f_3^*, \quad (3.7)
\]

at \( \xi = 1 \):

\[
\gamma_{42} \frac{1}{\mu} \left( \frac{du}{d\xi} + \alpha w - \psi \right) + \gamma_{41} u = \gamma_{41} f_4 + \gamma_{42} f_4^*, \quad (3.8)
\]

\[
\gamma_{52} \left( \frac{dw}{d\xi} - \alpha u \right) + \gamma_{51} w = \gamma_{51} f_5 + \gamma_{52} f_5^*, \quad (3.9)
\]

\[
\gamma_{62} \frac{d\psi}{d\xi} + \gamma_{61} \psi = \gamma_{61} f_6 + \gamma_{62} f_6^*, \quad (3.10)
\]

where

\[
\gamma_{11} = \frac{\beta_1}{1 + \beta_1}, \quad \gamma_{22} = \frac{1}{1 + \beta_1}, \quad i = 1, \ldots, 6 \quad (3.11)
\]

It should be noted that if the moment load \( m(\xi) \) is neglected and the beam is clamped at both ends, the coupled governing differential equations (3.2)-(3.4) and the boundary conditions (3.5)-(3.10) become the same as those given by Reddy and Volpi [11]. Taking the method given by Lee and Lin [19], the angle of rotation due to bending, and the transverse displacement can be expressed in terms of the tangential displacement, respectively

\[
\psi = \frac{1}{\alpha \delta_1} \left\{ \delta_2 \frac{d^4 w}{d\xi^4} + \delta_3 \frac{d^3 w}{d\xi^3} + \alpha^2 \delta_1 w + \left[ \frac{1}{\zeta} - \alpha^2 \mu \left( \frac{1}{\zeta} + \mu \right) \right] q + \delta_2 \left[ \alpha m + \alpha^2 \delta_2 \frac{dp}{d\xi} + \frac{1}{\zeta} \frac{d^2 q}{d\xi^2} \right] \right\},
\quad (3.12.1)
\]

\[
u = \frac{1}{\alpha^2 \delta_1} \frac{d^3 w}{d\xi^3} - \frac{1}{\alpha \zeta \delta_1} \frac{d^3 w}{d\xi^3} + \frac{1}{\alpha} \frac{dw}{d\xi} + \frac{1}{\alpha^2 \zeta} p - \frac{\delta_2}{\alpha^2 \zeta \delta_1} \frac{d^2 p}{d\xi^2} + \frac{1}{\alpha^2 \zeta \delta_2} \left[ \frac{1}{\zeta} \frac{d^2 q}{d\xi^2} - \frac{1}{\delta_1} \frac{dq}{d\xi} \right. \left. - \frac{1}{\alpha \delta_1} \left( \alpha \frac{dm}{d\xi} - \frac{1}{\zeta} \frac{d^2 q}{d\xi^2} \right) \right],
\quad (3.12.2)
\]
where
\[ \delta_1 = 1 + \alpha^2 \left( \mu + \frac{1}{\zeta} \right), \quad \delta_2 = \mu + \frac{1}{\zeta}, \quad \delta_3 = 1 + 2\alpha^2 \left( \mu + \frac{1}{\zeta} \right). \] (3.12.3)

Substituting the relations (3.12.1–3) into Eqs. (3.3) and (3.5)–(3.11), the complete sixth-order ordinary differential characteristic equation and the associated boundary conditions in the tangential displacement are obtained, respectively

\begin{equation}
\frac{d^6 w}{d \xi^6} + 2\alpha^2 \frac{d^4 w}{d \xi^4} + \alpha^4 \frac{d^2 w}{d \xi^2} = \alpha \frac{d p}{d \xi} - \left( \frac{1}{\zeta} + \mu \right) \frac{d^3 p}{d \xi^3} - \alpha^2 q + \alpha^2 \mu \frac{d^2 q}{d \xi^2} - \frac{1}{\zeta} \frac{d^4 q}{d \xi^4} - \alpha^2 m - \alpha \frac{d^2 m}{d \xi^2},
\end{equation}

(3.13)

at \( \xi = 0 \):

\begin{align}
\gamma_{11} \frac{d^6 w}{d \xi^6} - \gamma_{12} \alpha^2 \frac{d^4 w}{d \xi^4} + \gamma_{11} \alpha^2 \frac{d^3 w}{d \xi^3} - \gamma_{12} \alpha^2 \frac{d^2 w}{d \xi^2} - \gamma_{11} \alpha^2 \delta_1 \frac{d w}{d \xi} &= -\alpha^2 \delta_1 (\gamma_{11} f_1 + \gamma_{12} f_1^*), \\
\gamma_{22} \frac{d^6 w}{d \xi^6} + \gamma_{22} \alpha^2 \frac{d^4 w}{d \xi^4} - \gamma_{21} \alpha^2 \delta_1 w &= -\alpha^2 \delta_1 (\gamma_{21} f_2 + \gamma_{22} f_2^*), \\
\gamma_{32} \delta_2 \frac{d^6 w}{d \xi^6} - \gamma_{31} \delta_2 \frac{d^4 w}{d \xi^4} + \gamma_{32} \delta_3 \frac{d^3 w}{d \xi^3} - \gamma_{31} \delta_3 \frac{d^2 w}{d \xi^2} + \gamma_{32} \alpha^2 \delta_1 \frac{d w}{d \xi} - \gamma_{31} \alpha^2 \delta_1 w &= -\mu \delta_1 \left( \gamma_{31} f_3 + \gamma_{32} f_3^* \right),
\end{align}

(3.14–15, 16)

at \( \xi = 1 \):

\begin{align}
-\gamma_{41} \frac{d^6 w}{d \xi^6} - \gamma_{42} \alpha^2 \frac{d^4 w}{d \xi^4} - \gamma_{41} \alpha^2 \frac{d^3 w}{d \xi^3} - \gamma_{42} \alpha^2 \frac{d^2 w}{d \xi^2} + \gamma_{41} \alpha^2 \delta_1 \frac{d w}{d \xi} &= \alpha^2 \delta_1 \left( \gamma_{41} f_4 + \gamma_{42} f_4^* \right), \\
\gamma_{52} \delta_2 \frac{d^6 w}{d \xi^6} + \gamma_{52} \alpha^2 \frac{d^4 w}{d \xi^4} + \gamma_{52} \alpha^2 \delta_1 w &= \alpha^2 \delta_1 \left( \gamma_{52} f_5 + \gamma_{52} f_5^* \right), \\
\gamma_{62} \delta_2 \frac{d^6 w}{d \xi^6} + \gamma_{61} \delta_2 \frac{d^4 w}{d \xi^4} + \gamma_{62} \delta_3 \frac{d^3 w}{d \xi^3} + \gamma_{61} \delta_3 \frac{d^2 w}{d \xi^2} + \gamma_{62} \alpha^2 \delta_1 \frac{d w}{d \xi} + \gamma_{61} \alpha^2 \delta_1 w &= \mu \delta_1 \left( \gamma_{61} f_6 + \gamma_{62} f_6^* \right).
\end{align}

(3.17–19)
The six exact independent normalized homogeneous solutions of Eq. (3.13) are

\[ V_1(\xi) = 1, \]
\[ V_2(\xi) = \xi, \]
\[ V_3(\xi) = \frac{2}{\alpha^2} (1 - \cos \alpha \xi) - \frac{\xi}{2\alpha} \cos \alpha \xi, \]
\[ V_4(\xi) = \frac{2}{\alpha^2} \xi - \frac{5}{2\alpha^3} \sin \alpha \xi + \frac{\xi}{2\alpha^2} \cos \alpha \xi, \]
\[ V_5(\xi) = \frac{1}{\alpha^4} (1 - \cos \alpha \xi) - \frac{\xi}{2\alpha^3} \sin \alpha \xi, \]
\[ V_6(\xi) = \frac{\xi}{\alpha^4} - \frac{3}{2\alpha^3} \sin \alpha \xi + \frac{\xi}{2\alpha^4} \cos \alpha \xi, \]

which satisfy the following normalized condition

\[
\begin{bmatrix}
V_1(0) & V_2(0) & V_3(0) & V_4(0) & V_5(0) & V_6(0) \\
V_1'(0) & V_2'(0) & V_3'(0) & V_4'(0) & V_5'(0) & V_6'(0) \\
V_1''(0) & V_2''(0) & V_3''(0) & V_4''(0) & V_5''(0) & V_6''(0) \\
V_1'''(0) & V_2'''(0) & V_3'''(0) & V_4'''(0) & V_5'''(0) & V_6'''(0) \\
V_1^{(4)}(0) & V_2^{(4)}(0) & V_3^{(4)}(0) & V_4^{(4)}(0) & V_5^{(4)}(0) & V_6^{(4)}(0) \\
V_1^{(5)}(0) & V_2^{(5)}(0) & V_3^{(5)}(0) & V_4^{(5)}(0) & V_5^{(5)}(0) & V_6^{(5)}(0)
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

The general solution of Eq. (3.13) is

\[ w(\xi) = w_p(\xi) + \sum_{i=1}^{6} C_i V_i(\xi), \]

where the particular solution \( w_p(\xi) \) can be obtained from Eqs. (2.4), (2.6), and (2.13). Substituting the general solution (3.22) into the boundary conditions (3.14)-(3.19), the associated coefficients \( \{C_i\} \) are obtained

\[
\begin{bmatrix}
C_1 \\
C_2 \\
C_3 \\
C_4 \\
C_5 \\
C_6
\end{bmatrix}
= \begin{bmatrix}
b_{36} & b_{35} & b_{34} & b_{33} & b_{32} & b_{31} \\
b_{26} & b_{25} & b_{24} & b_{23} & b_{22} & b_{21} \\
\gamma_2 b_{12} & -\gamma_1 b_{12} & \gamma_3 b_{12} & -\gamma_1 b_{13} & \gamma_2 b_{12} & \gamma_1 b_{14} \\
\gamma_2 & 0 & \gamma_2 & 0 & 0 & -\gamma_2 \\
\gamma_1 & -\gamma_1 \alpha^2 & \gamma_1 \alpha^2 & -\gamma_1 \alpha^2 & \gamma_1 \alpha^2 & 0
\end{bmatrix}
^{-1}
\begin{bmatrix}
a_6 \\
a_5 \\
a_4 \\
a_3 \\
a_2 \\
a_1
\end{bmatrix}.
\]

where

\[
a_1 = - \left( \frac{\gamma_1}{\xi} \frac{d^2 w_p(0)}{d\xi^2} - \gamma_2 \alpha^2 \frac{d^4 w_p(0)}{d\xi^4} + \gamma_1 \alpha^2 \frac{d^3 w_p(0)}{d\xi^3} - \gamma_2 \alpha^2 \frac{d^2 w_p(0)}{d\xi^2} \right)
- \gamma_1 \alpha^2 \frac{d w_p(0)}{d\xi},
\]
\[
a_2 = - \left( \frac{\gamma_2}{\xi} \frac{d^2 w_p(0)}{d\xi^2} - \gamma_2 \alpha^2 \frac{d^4 w_p(0)}{d\xi^4} + \gamma_2 \alpha^2 \frac{d^3 w_p(0)}{d\xi^3} - \gamma_2 \alpha^2 \frac{d^2 w_p(0)}{d\xi^2} \right) - \gamma_2 \alpha \xi_2,
\]
\[
a_3 = - \left( \frac{\gamma_1}{\xi} \frac{d^2 w_p(0)}{d\xi^2} - \gamma_1 \alpha^2 \frac{d^4 w_p(0)}{d\xi^4} + \gamma_3 \alpha^2 \frac{d^3 w_p(0)}{d\xi^3} - \gamma_1 \alpha \frac{d^2 w_p(0)}{d\xi^2} \right)
+ \gamma_2 \alpha \frac{d w_p(0)}{d\xi} - \gamma_1 \alpha_2 \frac{d w_p(0)}{d\xi} - \gamma_3 \alpha \frac{d w_p(0)}{d\xi}
- \alpha \xi_3,
\]

\[ (3.24.1) \]
\[ a_4 = \frac{\gamma_{41}}{\zeta} \frac{d^2 w_p(1)}{dx^2} + \gamma_{42} \alpha^2 \frac{d^3 w_p(1)}{dx^3} + \gamma_{41} \alpha^2 \frac{d^3 w_p(1)}{dx^3} + \gamma_{42} \alpha^4 \frac{d^4 w_p(1)}{dx^4} - \gamma_{41} \delta_1 \alpha^2 \frac{dw_p(1)}{d\xi} + \alpha^3 \delta_1 \varepsilon_4, \]

\[ a_5 = -\left( \frac{\gamma_{52}}{\zeta} \frac{d^3 w_p(1)}{dx^3} + \gamma_{52} \alpha^2 \frac{d^3 w_p(1)}{dx^3} + \gamma_{51} \delta_1 \alpha^2 w_p(1) \right) + \alpha^2 \delta_1 \varepsilon_5, \] (3.24.1)

\[ a_6 = -\gamma_{62} \delta_2 \frac{d^2 w_p(1)}{d\xi^2} - \gamma_{61} \delta_2 \frac{d^2 w_p(1)}{d\xi^2} - \gamma_{62} \delta_3 \frac{d^2 w_p(1)}{d\xi^2} - \gamma_{61} \delta_3 \frac{d^2 w_p(1)}{d\xi^2} - \gamma_{62} \delta_2 \frac{d^2 w_p(1)}{d\xi^2} - \gamma_{61} \delta_1 \alpha^2 w_p(1) + \alpha \delta_1 \varepsilon_6; \]

\[ b_{1i} = -\frac{\gamma_{41}}{\zeta} \frac{d^2 V_i(1)}{d\xi^2} - \gamma_{42} \alpha^2 \frac{d^3 V_i(1)}{d\xi^3} - \gamma_{41} \alpha^2 \frac{d^3 V_i(1)}{d\xi^3} - \gamma_{42} \alpha^4 \frac{d^4 V_i(1)}{d\xi^4} + \gamma_{41} \delta_1 \alpha^2 \frac{dV_i(1)}{d\xi}, \] (3.24.2)

\[ b_{2i} = \gamma_{52} \frac{d^3 V_i(1)}{d\xi^3} + \gamma_{52} \alpha^2 \frac{d^3 V_i(1)}{d\xi^3} + \gamma_{51} \delta_1 V_i(1), \]

\[ b_{3i} = \gamma_{62} \delta_2 \frac{d^2 V_i(1)}{d\xi^2} + \gamma_{61} \delta_2 \frac{d^2 V_i(1)}{d\xi^2} + \gamma_{62} \delta_3 \frac{d^2 V_i(1)}{d\xi^2} + \gamma_{61} \delta_3 \frac{d^2 V_i(1)}{d\xi^2} + \gamma_{62} \delta_2 \frac{d^2 V_i(1)}{d\xi^2} + \gamma_{61} \delta_1 \alpha^2 V_i(1), \quad i = 1, \ldots, 6 \]

in which

\[ \delta_1 = 1 + \alpha^2 \left( \mu + \frac{1}{\zeta} \right), \quad \delta_2 = \mu + \frac{1}{\zeta}, \quad \delta_3 = 1 + 2 \alpha^2 \left( \mu + \frac{1}{\zeta} \right), \]

\[ \varepsilon_i = \gamma_{11} f_i + \gamma_{10} f_i^2, \quad i = 1, \ldots, 5, 6 \]

3.2 Inextensional curved Timoshenko beams

For inextensional beams the effect of centerline extensionsibility is neglected. By letting \( \zeta \to \infty \) in Eq. (3.13), the governing characteristic differential equation is obtained:

\[ \frac{d^8 w}{dx^8} + 2\alpha^2 \frac{d^4 w}{dx^4} + \alpha^4 \frac{d^4 w}{dx^4} = \alpha \frac{dp}{d\xi} - \mu \frac{d^3 p}{d\xi^3} - \alpha^2 q + \alpha^2 \mu \frac{d^2 q}{d\xi^2} - \alpha^3 m - \alpha \frac{d^2 m}{d\xi^2}. \] (3.26)

The relations between the angle of rotation due to bending, the transverse displacement and the tangential displacement are obtained via Eqs. (3.12.1–3), respectively

\[ \psi = \frac{1}{\alpha^2 (\alpha^2 \mu + 1)} \left\{ \alpha \mu \frac{d^4 w}{d\xi^4} + \alpha(1 + 2\alpha^2 \mu) \frac{d^2 w}{d\xi^2} + \alpha^3(1 + \alpha^2 \mu) w - \alpha^3 \mu^2 q + \alpha \mu \left( am + \alpha \mu \frac{dp}{d\xi} \right) \right\}, \] (3.27)

\[ u = \frac{1}{\alpha} \frac{dw}{d\xi}. \] (3.28)

The corresponding general solution can be obtained via Eq. (3.22) by letting \( \zeta \to \infty \).

4 Deflection curves and exact element stiffness matrix

To establish the element stiffness matrix, the deflection curves due to a unit generalized displacement at one of the nodal coordinates are to be determined. Moreover, the consistent mass matrix can be constructed via the curves. A finite element method can be successfully developed for the dynamic analysis [18].
Letting $\gamma_{i1} = 1$, and $\gamma_{i2} = f_{ji}^* = 0$, $i = 1, 2, \ldots, 6$, and $f_{n} = 1$ while all other nodal coordinates are maintained at zero generalized displacement, $n = 1, 2, \ldots, 6$, respectively, six kinds of the boundary conditions are obtained as following:

- **case 1:** $u(0) = 1$, $w(0) = \Psi(0) = u(1) = w(1) = \Psi(1) = 0$;
- **case 2:** $w(0) = 1$, $u(0) = \Psi(0) = u(1) = w(1) = \Psi(1) = 0$;
- **case 3:** $\Psi(0) = 1$, $u(0) = w(0) = u(1) = w(1) = \Psi(1) = 0$; (4.1.1 – 6)
- **case 4:** $u(1) = 1$, $u(0) = w(0) = \Psi(0) = \Psi(1) = \Psi'(0) = \Psi'(1) = 0$;
- **case 5:** $w(1) = 1$, $u(0) = w(0) = \Psi(0) = u(1) = \Psi(1) = 0$;
- **case 6:** $\Psi(1) = 1$, $u(0) = w(0) = \Psi(0) = u(1) = w(1) = 0$.

Neglecting the external forces and moments $p(\xi)$, $q(\xi)$ and $m(\xi)$, and substituting one of the six kinds of conditions into the general solution (3.22), the corresponding displacement $w(\xi)$ is obtained

$$w(\xi) = \sum_{i=1}^{6} C_i V_i(\xi),$$

where the corresponding coefficients are listed in the Appendix. Substituting $w(\xi)$ into the relations (3.12.1–3), respectively, the displacements $u(\xi)$ and the angle due to bending $\Psi(\xi)$ are obtained.

Letting the external forces and moment be zero and based on Eq. (3.3) and the relations (3.12.1–3), the dimensionless normal force $N(\xi)$, the bending moment $M(\xi)$ and the shear force $Q(\xi)$ can be expressed in terms of the tangential displacement $w$, respectively

$$N(\xi) = \zeta \alpha \left( \frac{dw}{d\xi} - \alpha u \right) = -\frac{1}{\alpha^2 \delta_1} \frac{d^2 w}{d\xi^2} + \frac{1}{\delta_1} \frac{d^3 w}{d\xi^3},$$

$$Q(\xi) = \frac{\alpha}{\mu} \left( \frac{dw}{d\xi} - \Psi \right) = -\frac{1}{\alpha^3 \delta_1} \frac{d^2 w}{d\xi^2} + \frac{1}{\alpha \delta_1} \frac{d^4 w}{d\xi^4},$$

$$M(\xi) = -\frac{d\Psi}{d\xi} = \frac{-1}{\alpha \delta_1} \left[ \delta_2 \frac{d^2 w}{d\xi^2} + \delta_3 \frac{d^3 w}{d\xi^3} + \alpha^2 \delta_1 \frac{d w}{d\xi} \right].$$

The relation between the generalized forces and displacements at the nodal coordinate is written in matrix notation as

$$\begin{bmatrix} N(0) \\ Q(0) \\ M(0) \\ N(1) \\ Q(1) \\ M(1) \end{bmatrix} = [k_{ij}]_{6\times6} \begin{bmatrix} u(0) \\ w(0) \\ \Psi(0) \\ u(1) \\ w(1) \\ \Psi(1) \end{bmatrix}.$$  (4.4)

Substituting the six kinds of deflection curves (4.2) due to a unit generalized displacement at one of the nodal coordinates into Eqs. (4.3.1–3), the stiffness matrix $[k_{ij}]$ can be obtained

$$k_{1j} = \sum_{i=1}^{6} \frac{T_{i(7-j)}}{\delta_1} \left[ -\frac{1}{\alpha^2 \delta_1} \frac{d^2 V_i(0)}{d\xi^2} + \frac{d^3 V_i(0)}{d\xi^3} \right],$$

$$k_{2j} = \sum_{i=1}^{6} \frac{T_{i(7-j)}}{\alpha \delta_1} \left[ -\frac{1}{\alpha^2 \delta_1} \frac{d^3 V_i(0)}{d\xi^3} + \frac{d^4 V_i(0)}{d\xi^4} \right],$$

$$k_{3j} = \sum_{i=1}^{6} \frac{T_{i(7-j)}}{\alpha \delta_1} \left[ \delta_2 \frac{d^2 V_i(0)}{d\xi^2} + \delta_3 \frac{d^3 V_i(0)}{d\xi^3} + \alpha^2 \delta_1 \frac{d V_i(0)}{d\xi} \right].$$
\[ k_{ij} = \sum_{i=1}^{6} \frac{T_{i}(\tau-j)}{\delta_{1}} \left[ \frac{1}{\alpha^{2}} \frac{d^{5}V_{i}(1)}{d\xi^{5}} + \frac{d^{3}V_{i}(1)}{d\xi^{3}} \right], \]
\[ k_{5j} = \sum_{i=1}^{6} \frac{T_{i}(\tau-j)}{\alpha\delta_{1}} \left[ -\frac{1}{\alpha^{2}} \frac{d^{5}V_{i}(1)}{d\xi^{5}} + \frac{d^{3}V_{i}(1)}{d\xi^{3}} \right], \]
\[ k_{5j} = \sum_{i=1}^{6} \frac{T_{i}(\tau-j)}{\alpha\delta_{1}} \left[ \epsilon_{2} \frac{d^{5}V_{i}(1)}{d\xi^{5}} + \epsilon_{3} \frac{d^{3}V_{i}(1)}{d\xi^{3}} + \alpha^{2} \frac{dV_{i}(1)}{d\xi} \right]. \]

5 Verification

To verify the previous analysis, the following example is presented.

**Example:** Consider the deformation of a cantilever extensional Timoshenko beam subjected to a unit transverse load at the right end of the beam. The corresponding coefficients are

\[
\gamma_{11} = \gamma_{21} = \gamma_{31} = \gamma_{42} = \gamma_{52} = \gamma_{62} = 1, \quad \gamma_{12} = \gamma_{22} = \gamma_{32} = \gamma_{41} = \gamma_{51} = \gamma_{61} = 0,
\]

\[
p(\xi) = q(\xi) = m(\xi) = f_{1} = f_{1}^{*} = f_{2} = f_{2}^{*} = f_{3} = f_{3}^{*} = f_{4} = f_{5} = f_{6} = f_{6}^{*} = 0, \quad (5.1)
\]

\[
f_{4}^{*} = 1, \quad \alpha = \pi/2.
\]

The homogeneous solutions (3.20) have been obtained. The particular solution is zero. Accordingly, the corresponding general solution in the tangential displacement can be obtained via Eq. (3.22). Substituting the general solution into Eqs. (3.12.1–3), the solutions in transverse displacement and the angle of rotation due to bending are obtained, respectively. The solution in transverse displacement determined by using Castigliano’s second

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<td>0.212 64</td>
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</tr>
</tbody>
</table>

* Present study
** By using Castigliano’s second theorem
theorem is

\[
w(\xi) = \left(\frac{1}{\pi^3} + \frac{1}{\pi^5}\right) \left[ \sin^3\left(\frac{\pi \xi}{2}\right) - \cos\left(\frac{\pi \xi}{2}\right) \left(\frac{\pi \xi}{4} - \sin\pi \xi\right) \right] - \frac{\mu}{\pi} \left[ \sin^3\left(\frac{\pi \xi}{2}\right) + \cos\left(\frac{\pi \xi}{2}\right) \left(\frac{\pi \xi}{4} + \sin\pi \xi\right) \right],
\]

(5.2)

It is shown in Table 1 that the numerical results given by the present analysis are the same as those determined by using Castigliano’s second theorem.

6 Conclusion

In this paper, a generalized Green function of a nth-order ordinary differential equation with forcing function composed of the delta function and its derivatives is obtained. The generalized Green function can be easily and effectively applied to both the boundary value problems and the initial value problems. It is the generalization of those given by Pan and Hohenstein, and Kanwal. The exact solution for static analysis of an extensible circular curved Timoshenko beam with nonhomogeneous elastic boundary conditions is obtained based on the generalized Green function. The explicit relations between the angle of rotation due to bending, the transverse displacement and the tangential displacement of the beam, the deflection curves due to a unit generalized displacement at nodal coordinate, and the exact element stiffness matrix are obtained. A finite element method can be developed based on the results for the dynamic analysis. Meanwhile, the stiffness locking phenomena accompanied in some other curved beam element methods does not exist in the proposed method.

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Appendix: The coefficients of deflection curves

Case 1: The corresponding solution \(w(\xi)\) is

\[
w(\xi) = \sum_{i=1}^{6} T_{10} V_i(\xi),
\]

where

\[
[T_{ij}] = \begin{bmatrix}
 b_{26} & b_{25} & b_{24} & b_{31} & b_{31} & b_{31} \\
b_{26} & b_{25} & b_{24} & b_{23} & b_{22} & b_{21} \\
b_{16} & b_{15} & b_{14} & b_{13} & b_{12} & b_{11} \\
0 & -\delta_2 & 0 & -\delta_3 & 0 & -\alpha^2 \delta_1 \\
0 & 0 & 0 & 0 & 0 & -\alpha^2 \delta_1 \\
\frac{1}{\xi} & 0 & \frac{\alpha^2}{\xi} & 0 & -\alpha^2 \delta_1 & 0
\end{bmatrix}^{-1},
\]
in which
\[ b_{i2} = -\frac{1}{\zeta} \frac{d^2 V_i(1)}{d\xi^2} - \alpha^2 \frac{d^2 V_i(1)}{d\xi^2} + \delta_1 \alpha^2 \frac{d V_i(1)}{d\xi}, \]
\[ b_{i3} = \alpha^2 \delta_2 V_i(1), \]
\[ b_{i4} = \delta_2 \frac{d^4 V_i(1)}{d\xi^4} + \delta_3 \frac{d^2 V_i(1)}{d\xi^2} + \delta_1 \alpha^2 V_i(1), \quad i = 1, \ldots, 6 \]
\[ \delta_1 = 1 + \alpha^2 \left( \mu + \frac{1}{\zeta} \right), \quad \delta_2 = \mu + \frac{1}{\zeta}, \quad \delta_3 = 1 + 2\alpha^2 \left( \mu + \frac{1}{\zeta} \right). \]

Case 2: The corresponding solution \( w(\xi) \) is
\[ w(\xi) = \sum_{i=1}^{6} T_{i5} V_i(\xi), \]
where the coefficients are the same as those of case 1.

Case 3: The corresponding solution \( w(\xi) \) is
\[ w(\xi) = \sum_{i=1}^{6} T_{i4} V_i(\xi), \]
where the coefficients are the same as those of case 1.

Case 4: The corresponding solution \( w(\xi) \) is
\[ w(\xi) = \sum_{i=1}^{6} T_{i3} V_i(\xi), \]
where the coefficients are the same as those of case 1.

Case 5: The corresponding solution \( w(\xi) \) is
\[ w(\xi) = \sum_{i=1}^{6} T_{i2} V_i(\xi), \]
where the coefficients are the same as those of case 1.

Case 6: The corresponding solution \( w(\xi) \) is
\[ w(\xi) = \sum_{i=1}^{6} T_{i1} V_i(\xi), \]
where the coefficients are the same as those of Case 1.

References
Exact solutions for curved Timoshenko beams


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