Nonlocal elasticity effect on column buckling of multiwalled carbon nanotubes under temperature field

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ABSTRACT

Based on nonlocal theory of thermal elasticity mechanics, an elastic multiple column model is developed for column buckling of MWNs with large aspect ratios under axial compression coupling with temperature change. The present model treats each of the nested tubes as an individual column interacting with adjacent nanotubes through the intertube van der Waals forces. The thermal effect is incorporated in the formulation. In particular, an explicit expression is derived for the critical axial strain of a double-walled carbon nanotube which clearly demonstrates that small scale effects contribute significantly to the thermo-mechanical behavior of multiwalled carbon nanotubes and cannot be ignored.

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1. Introduction

Carbon nanotubes (CNTs) are cylindrical macromolecules consisted of carbon atoms in a periodic hexagonal structure. Research on the mechanical properties of carbon nanotubes has been proposed since CNTs were discovered by Iijima [1]. The results from the research show that CNTs exhibit superior mechanical properties. Although there are various reports in the literature on the exact properties of CNTs, theoretical and experimental results have shown an extremely high elastic modulus, greater than 1 TPa (the elastic modulus of diamond is 1.2 TPa), for CNTs. Reported strengths of CNTs are 10–100 times higher than the strongest steel at a fraction of the weight. Thus, mechanical behavior of CNTs has been the subject of numerous recent studies [2–14].

The modelling for the analytical analysis of CNTs is mainly classified into two categories. The first one is the atomic modelling, including the techniques such as classical molecular dynamics (MD) [15,16], tight binding molecular dynamics (TBMD) [17] and density functional theory (DFT) [18], which is only limited to systems with a small number of molecules and atoms and therefore only restrained to the study of small-scale modelling. On the other hand, continuum modelling is practical in analyzing CNTs with large-scale sizes. Yakobson et al. [19] studied axially compressed buckling of single-walled carbon nanotubes using molecular dynamics simulations. These authors compared their simulation results with a simple continuum shell model and found that all changes in buckling pattern can be predicted using a continuum model.

Although the classic continuum models are relevant to some extent, and efficient in computation for models at large length scales, the applicability of these classical continuum models at small length scales is questionable. The key issue is that at small length scales the material microstructure (such as lattice spacing between individual atoms) becomes increasingly important and the discrete structure of the material can no longer be homogenized into a continuum [20]. On the other hand, molecular models, while conceptually valid for small length scales, are difficult to formulate accurately and are almost
always computationally intensive and prohibitively expensive. Even when a solution is found, it has to be compared with
ones obtained by experimental investigations. A possible solution to the aforementioned difficulty is extending the contin-
uum approach to smaller length scales by incorporating information regarding the behavior of the material microstructure.
This is accomplished quite readily by adopting the theory of nonlocal continuum mechanics.

In classical (local) elasticity theory the stress tensor at a given point depends linearly on the strain tensor of the same
point. Thus, local elastic theory contains no information about the long range forces between atoms (i.e., there is no internal
length scale). On the other hand, the theory of nonlocal continuum mechanics assumes that the stress state at a given
reference point is considered to be function of the strain states of all points in the body. In this way, the internal length scale
enters into the constitutive equations simply as a material parameter. A recent study made reference to the fact that “non-
local continuum mechanics could potentially play a useful role in analysis related to nanotechnology applications” [21].

It is known that the phenomenon of buckling often occurs when CNTs are subjected to compressive loads. Thus, buckling
of CNTs has become one of the topics of primary interest [22–24], and many continuum buckling models have been devel-
oped [9,19,25–32]. Yakobson et al. [19] presented a continuum shell model in studying axially compressed buckling of sin-
gle-walled nanotubes (SWNTs). Ru [25] developed a multiple-elastic beam model to study column buckling of multiwalled
carbon nanotubes (MWNTs) embedded within an elastic medium. Based on a multiple-shell model [26], Wang et al. [27]
investigated axially compressed buckling of MWNTs under radial pressure. Sudak [28] discussed column buckling of MWNTs
on the basis of nonlocal continuum mechanics. Zhang et al. [29] studied bending instability characteristics of double-walled
carbon nanotubes (DWNTs) of various configurations.

Lately, much research indicates that the mechanical properties of CNTs are related to temperature change. Raravikar et al.
[33] studied the temperature dependence of radial breathing mode Raman frequency of SWNTs by using MD simulation and
found that the coefficients of thermal expansion are positive in both radial and axial directions as the temperature is varied
from 300 to 800 K. Schelling and Kebinski [34] obtained similar results through MD simulation. Pipes and Hubert [35] inves-
tigated thermal expansion of helical CNTs arrays, and the effective axial, transverse and shearing coefficients of thermal
expansion of the array are determined. Based on the interatomic potential and the local harmonic model, Jiang et al. [36]
presented an analytical method to determine the coefficient of thermal expansion for SWNTs. They concluded that all the
coefficients of thermal expansion are negative at low and room temperature and become positive at high temperature. Con-
sequently, the investigation of thermal effect on the mechanical properties of CNTs is of great importance and necessity.
However, all analyses of buckling for CNTs mentioned above have not accounted for the thermal effect. Very lately, Yao
and Han [37] conducted buckling analysis of MWNTs subjected to torsional load under temperature field. Based on a rigor-
ous van der Waals interaction, Wang et al. [38] conducted an investigation on the axially compressed buckling of MWNTs
under thermal loads via a continuum shell model. On the basis of theory of thermal elasticity mechanics, an elastic multiple
column model is developed by Zhang et al. [39] for column buckling of MWNTs with large aspect ratios under axial compres-
sion coupling with temperature change, which takes into account the effect of temperature change in the formulation.

As far as we know, there has been no investigation on thermal effect on column buckling of MWNTs using the nonlocal
beam model. In this paper, based on nonlocal theory of thermal elasticity mechanics, an elastic multiple column model is
developed for the linearized column buckling of MWNTs with large aspect ratios under axial compression coupling with
temperature change and small scale effect. The effects of temperature change and the small length scale on the properties
of axially column buckling is examined.

2. Nonlocal multiple column model with thermal effect

Multiwalled carbon nanotubes are distinguished from traditional elastic beams by their hollow multilayer structure and
associated intertube van der Waals forces. In fact, there is strong compelling evidence that suggests that the van der Waals
forces between adjacent tubes have a crucial effect on the mechanical behavior of carbon nanotubes [26]. Since axially com-
pressed buckling of carbon nanotubes represents a basic mechanical property, the effects of small length scales and temper-
ature change on the axial buckling of the intertube van der Waals forces is of great interest. Motivated by this, a nonlocal
elastic column model is presented for axially compressed buckling of multiwalled carbon nanotubes.

The fundamental assumption behind the Euler–Bernoulli beam bending model is that the beam consists of fibers which
are in a state of uniaxial compression or tension. Assuming the linear form of Hooke’s law, it is well known that the deflection
curve of an elastic column under constant axial load and distributed lateral pressure is given by the classic result [40]

\[
\frac{N}{E} \frac{d^2 w}{dx^2} + p(x) = \frac{EI}{\alpha^4} \frac{d^4 w}{dx^4},
\]

where \( x \) is the axial coordinate, \( N \) is the constant axial force, \( p(x) \) is the distributed lateral pressure per unit length (measured
positive in the direction of the deflection), \( EI \) is the flexural stiffness of the column, and \( w(x) \) is the deflection. It has been
reported in a recent study that the classical elastic models can be applied safely to investigate the mechanical behavior of
carbon nanotubes when the length scales are about 100 nm and above [41]. However, at very small length scales, these clas-
sical continuum models overlook the discrete material makeup of the material and inevitably break down. A possible solu-
tion is utilizing nonlocal continuum mechanics which represents an attempt to recognize the finite range of interatomic and
intermolecular forces.
For the axial force \( N \), we have
\[
N = \sigma A = N_m + N_t, \tag{2}
\]
where \( A \) is the cross-sectional area of the beam, and
\[
N_m = \sigma_m A, \quad N_t = -\frac{EA}{1-2\nu} \alpha \theta, \tag{3}
\]
where \( \sigma_m \) is the axial stress due to the mechanical loading prior to buckling, \( \alpha \) denotes the coefficient of thermal expansion in the direction of the \( x \)-axis, \( \theta \) is the temperature change, and \( E \) and \( \nu \) are Young’s modulus and Poisson’s ratio, respectively.

Substituting Eq. (2) into Eq. (1), we obtain
\[
(N_m + N_t) \frac{d^2 w}{dx^2} + p(x) = E I \frac{d^4 w}{dx^4}. \tag{4}
\]
Adopting the theory of nonlocal elasticity, Hooke’s law for a uniaxial stress state is determined by [11,21,28]
\[
\sigma = (e_0 a)^2 \frac{\partial^2 \sigma}{\partial x^2} = E \varepsilon, \tag{5}
\]
where \( e_0 \) is a constant that is appropriate to the material, for example, Eringen [42] gives \( e_0 = 0.39 \). However, it should be noted that the value of \( e_0 \) needs to be determined from experiments or by matching dispersion curves of plane waves with those of atomic lattice dynamics [42]. This has not yet been achieved for carbon nanotubes and currently remains an unresolved problem. However, based on the current study, the author speculates that in order for the nonlocal effect to have significance the value of \( e_0 \) should assume a value on the order of hundreds or even thousands. In addition, the parameter \( a \) describes the internal characteristic length (such as lattice spacing, granular distance); for example, for a single-walled carbon nanotube, the parameter \( a \) can be suitably chosen to be the length of a \( C-C \) bond which is found to be 0.142 nm [43].

For the case where the thermal effect is taken into account, Eq. (5) becomes
\[
\sigma = (e_0 a)^2 \frac{\partial^2 \sigma}{\partial x^2} = E \varepsilon - \frac{E}{1-2\nu} \alpha \theta, \tag{6}
\]
where \( \sigma \) is the axial stress, and \( \varepsilon \) is the axial strain.

To derive the nonlocal deflection curve of an elastic column that will model the buckling of carbon nanotubes, we make use of the following relations [40]:
\[
\frac{dV}{dx} = -p(x), \quad V = \frac{dM}{dx} + N \frac{dw}{dx}, \tag{7}
\]
where \( V(x) \) is the resultant shear force and \( M(x) \) is the resultant bending moment. Using the fact that \( M(x) = \int y \sigma(x) dA \) and noting that the relationship between strain and curvature for small deflections is \( \varepsilon = y/R \) and that \( 1/R = -(d^2 w/dx^2) \), where \( R \) is the radius of curvature and \( y \) is the coordinate measured positive in the direction of deflection [40], Eq. (6) can be rewritten as follows:
\[
M = (e_0 a)^2 \frac{\partial^2 M}{\partial x^2} - E I \frac{\partial^2 w}{\partial x^2}. \tag{8}
\]
Differentiating Eq. (8) twice and substituting Eq. (7) into the resulting expression the following expression can be derived for the nonlocal deflection curve of an elastic column under constant axial load and distributed lateral pressure:
\[
EI \frac{d^4 w}{dx^4} = p(x) + (N_m + N_t) \frac{d^2 w}{dx^2} - (e_0 a)^2 \left( \frac{d^2 p(x)}{dx^2} + (N_m + N_t) \frac{d^2 w}{dx^2} \right). \tag{9}
\]
Note that when parameter \( e_0 \) in Eq. (9) is set to zero, the classical Euler–Bernoulli expression with thermal effect is recovered. Here, let us assume that the lateral pressure \( p(x) \) is a continuous function of the axial coordinate \( x \).

As CNTs have high thermal conductivity, it may be regarded that the change of temperature is uniformly distributed in the CNT. Treacy et al. [44] reported a linear relationship between the mean-square vibration amplitude of the tube’s free tip displacement and the tube temperature, which implies that the tube’s elastic modulus is temperature independent. Hsieh et al. [45] studied the variation of Young’s modulus of SWNTs with temperature, and it was indicated that the Young’s modulus of an SWNT is insensitive to temperature change in the tube at temperatures of less than approximately 1100K, but decreases at higher temperatures. Thus, for the cases of low temperatures and high temperatures (but not very high), the Young’s modulus is herein assumed to be temperature independent. In what follows, all nested tubes are supposed to have the same thickness and effective material constants. By using the same steps of Sun and Liu [46,47], applying Eq. (9) to each layer of a MWNT under buckling, \( n \) coupled equations can be obtained as
respectively, and van der Waals pressure per unit axial length on tube yields

\[ C_k \]

where \( c_{k(k+1)} \) denotes the intertube interaction coefficient, which can be estimated by

\[ c_{k(k+1)} = \frac{320(2R)_{\text{erg/cm}^2}}{0.16a^2} \quad (k = 1, 2, \ldots, n - 1), \]

where \( R \) (measured in cm) is the inner radius of each pair of nanotubes.

Substitution of Eq. (11) into Eq. (10) gives

\[ \begin{align*}
E_l \frac{d^4w_l}{dx^4} &= c_{12}(w_2 - w_1) + (N_{m1} + N_{t1}) \frac{d^2w_1}{dx^2} - (e_0a)^2 \left( c_{12} \frac{d^2(w_2 - w_1)}{dx^2} + (N_{m1} + N_{t1}) \frac{d^2w_1}{dx^4} \right), \\
E_k \frac{d^4w_k}{dx^4} &= c_{k(k+1)}(w_{k+1} - w_k) - c_{k-1(k+1)}(w_k - w_{k-1}) + (N_{m(k+1)} + N_{t(k+1)}) \frac{d^2w_k}{dx^2} - (e_0a)^2 \left( \frac{d^2(w_{k+1} - w_k)}{dx^2} \right) - c_{k(k-1)} \frac{d^2(w_k - w_{k-1})}{dx^2} \right) \quad (k = 2, 3, \ldots, n - 1), \\
E_n \frac{d^4w_n}{dx^4} &= -c_{(n-1)n}(w_n - w_{n-1}) + (N_{m(n-1)} + N_{t(n-1)}) \frac{d^2w_n}{dx^2} - (e_0a)^2 \left( -c_{(n-1)n} \frac{d^2(w_n - w_{n-1})}{dx^2} + (N_{m(n-1)} + N_{t(n-1)}) \frac{d^4w_n}{dx^4} \right),
\end{align*} \]

where

\[ N_{m(k+1)} = \sigma_m A_k, \quad N_{t(k+1)} = -\frac{Ea_k}{1 - 2v}x_k \theta \quad (k = 1, 2, \ldots, n). \]

It is seen that these equations are coupled to each other due to the van der Waals interaction terms. In addition, it can be observed that with the small scale effect ignored Eqs. (13) reduce to the result

\[ \text{[39]} \]

3. Nonlocal buckling analysis

For simplification and without loss of generality, the nonlocal axial column buckling of a DWNT with a large aspect ratio is considered. In this case, Eqs. (13) become

\[ \begin{align*}
E_l \frac{d^4w_l}{dx^4} &= c_{12}(w_2 - w_1) + \left( \sigma_m - \frac{E}{1 - 2v}x_k \theta \right) A_1 \frac{d^2w_1}{dx^2} - (e_0a)^2 \left( c_{12} \frac{d^2(w_2 - w_1)}{dx^2} + \left( \sigma_m - \frac{E}{1 - 2v}x_k \theta \right) A_1 \frac{d^2w_1}{dx^4} \right), \\
E_k \frac{d^4w_k}{dx^4} &= -c_{12}(w_2 - w_1) + \left( \sigma_m - \frac{E}{1 - 2v}x_k \theta \right) A_2 \frac{d^2w_2}{dx^2} - (e_0a)^2 \left( -c_{12} \frac{d^2(w_2 - w_1)}{dx^2} + \left( \sigma_m - \frac{E}{1 - 2v}x_k \theta \right) A_2 \frac{d^2w_2}{dx^4} \right).
\end{align*} \]

(15a)

(15b)

Let us consider the hinged boundary conditions. For this case, we have

\[ w_1 = C \sin \left( \frac{m \pi}{L} x \right), \quad w_2 = D \sin \left( \frac{m \pi}{L} x \right), \]

where \( C \) and \( D \) are real constants, and \( m \) is a positive integer which is related to buckling modes. Introduction of Eqs. (16) into Eqs. (15a) and (15b) yields
Nontrivial solutions for the constants $C$ and $D$ exist only when the determinant of the coefficients in Eqs. (17a) and (17b) vanishes. In this manner, we have

$$X\sigma_m^2 + Y\sigma_m + Z = 0,$$

where

$$X = A_1 A_2 \left(\frac{m_1}{L}\right)^4 \left[1 + (e_0 a)^2 \left(\frac{m_1}{L}\right)^2\right]^2,$$

$$Y = E(I_1 A_2 + I_2 A_1) \left(\frac{m_1}{L}\right)^6 \left[1 + (e_0 a)^2 \left(\frac{m_1}{L}\right)^2\right] + \left[c_{12}(A_1 + A_2) \left(\frac{m_1}{L}\right)^2 - \frac{2E}{(1-2\nu)} \kappa_\theta A_1 A_2 \left(\frac{m_1}{L}\right)^4\right] \left[1 + (e_0 a)^2 \left(\frac{m_1}{L}\right)^2\right],$$

$$Z = E^2 I_1 I_2 \left(\frac{m_1}{L}\right)^8 + c_{12} E (I_1 + I_2) \left(\frac{m_1}{L}\right)^4 \left[1 + (e_0 a)^2 \left(\frac{m_1}{L}\right)^2\right] + \frac{E}{(1-2\nu)} \kappa_\theta \left[1 + (e_0 a)^2 \left(\frac{m_1}{L}\right)^2\right]$$

$$\times \left[\frac{E}{(1-2\nu)} \kappa_\theta A_1 A_2 \left(\frac{m_1}{L}\right)^4 \left[1 + (e_0 a)^2 \left(\frac{m_1}{L}\right)^2\right]\right].$$

In consequence, the critical axial buckling strain can be obtained by

$$\frac{-\sigma_m}{E} = -\sqrt{\left(\frac{24}{E}\right)^2 (A_1 + A_2)^2 \left[1 + (e_0 a)^2 \left(\frac{m_1}{L}\right)^2\right]^2 + \frac{2c_{12}}{A_1 A_2} \left(\frac{m_1}{L}\right)^2 (I_1 A_2 - I_2 A_1) \left(\frac{m_1}{L}\right)^4 \times \left[1 + (e_0 a)^2 \left(\frac{m_1}{L}\right)^2\right] + \frac{(I_1 A_2 + I_2 A_1) (\frac{m_1}{L})^4}{2A_1 A_2 (\frac{m_1}{L})^2} \left[1 + (e_0 a)^2 \left(\frac{m_1}{L}\right)^2\right] - \frac{\kappa_\theta}{1-2\nu}.}$$

When the effect of temperature change is ignored, the nonlocal result for the critical axial buckling strain [28] is recovered.

### 4. Results and discussion

On the basis of the above equations, we investigate the effect of the nonlocal parameter and temperature change on the axial buckling strain of a DWNT with numerical examples. As previously mentioned Jiang et al. [36] found that the coefficients of thermal expansion for CNTs are negative at low or room temperature and become positive at high temperature. The parameters used in calculations for the DWNT are given as follows: the Young’s modulus $E = 1$ TPa with the effective thickness of single-walled carbon nanotubes taken to be $t = 0.35$ nm [11]. The inner diameter $d_1 = 0.7$ nm and the outer diameter $d_2 = 1.4$ nm.

To illustrate the influence of temperature change on the nonlocal axial buckling strain of a DWNT, let us consider the difference $\delta$ between the critical axial buckling strain with the thermal effect included and that without the thermal effect. It follows from Eq. (19b) that

$$\delta = \left(\frac{-\sigma_m}{E}\right)_{th} - \left(\frac{-\sigma_m}{E}\right)_{nth} = \frac{\kappa_\theta}{1-2\nu},$$

where $\left(\frac{-\sigma_m}{E}\right)_{th}$ represents the critical buckling strain with the thermal effect and $\left(\frac{-\sigma_m}{E}\right)_{nth}$ denotes that without the thermal effect ($\theta = 0$). It can be observed from Eq. (19c) that the difference $\delta$ is linearly proportional to the temperature change $\theta$ of the nanotube.

To further investigate the effect of scale parameter and temperature change on the critical buckling strain, the following ratios are used:

$$\gamma_N = \left(\frac{-\sigma_m}{E}\right)_N / \left(\frac{-\sigma_m}{E}\right)_L, \quad \gamma_{th} = \left(\frac{-\sigma_m}{E}\right)_{th} / \left(\frac{-\sigma_m}{E}\right)_{nth},$$

where $\left(\frac{-\sigma_m}{E}\right)_N$ stands for the critical buckling strain based on the nonlocal beam model and $\left(\frac{-\sigma_m}{E}\right)_L$ denotes that based on local beam model.
4.1. Discussion on nonlocal parameter $e_0$

The magnitude of the nonlocal parameter, $e_0$, determines the nonlocal effect in the analysis. As defined by Eringen [42], $e_0$ is a constant appropriate to each material. For example, for a certain class of materials, by comparing the results of lattice dynamics with nonlocal theory, it was found that $e_0 = 0.39$. According to the Sudak [28], values of $e_0$ need to be determined from experimental results, which for SWCNTs are scarce. On the basis of results presented in [28], it was concluded that $L/a$ and $e_0$ be of the same order or one order less to have any significant nonlocal effect. Zhang et al. [8] estimated the value of $e_0$ by curve fitting the theoretical results obtained using nonlocal elasticity to those from MM simulations for the critical axial buckling strain of SWCNT. MM results obtained by Sears and Batra [48], were compared to the obtained for a nanotube modelled as a nonlocal elastic cylindrical shell using Donnell theory and the value of $e_0$ was approximated as 0.82. Wang et al. [30] used the value $0 < e_0 < 7$ for their analysis, which is much higher than values predicted earlier. Zhang et al. [49] determined the values of $e_0$ for different chiral angles, $(m, n)$, by curve fitting the results obtained by MD simulations and nonlocal analysis results obtained using their Donnell shell model. Values obtained varied from a minimum of 0.546 for a $(15, 4)$ chiral shell to a maximum of 1.043 for a $(11, 9)$ chiral shell. Zhang et al. [50] performed analysis of elastic interaction between Stone-Wales and divacancy defects on carbon graphene sheet. They concluded that the displacement field around defects obtained from the nonlocal continuum model and MDSs can match very well if $e_0$ is chosen to be 8.79. Wang and Hu [51], who adopted the second-order strain gradient constitutive relation, proposed that $e_0 = 1/\sqrt{12} = 0.288$ be used in the determination of the dispersion curves via elastic beam theories and the MD method. They obtained their $e_0$ value by a comparative study of the gradient method with atomic lattice dynamics of a one-dimensional lattice. Fig. 1 shows the dispersion of a crystal for the one-dimensional lattice, in which $e_0$ is the equivalent sound velocity in the crystal, and $c$ is the phase velocity in the lattice which is the ratio of the frequency $\omega$ to the wave number $k$ of the wave in the lattice. It can be seen from the figure that the results obtained using the gradient method are in excellent agreement with the dispersion curves obtained via the Born–Karman model of lattice dynamics at smaller values of $ka$.

Therefore, it can be concluded that the adopted value of the coefficient $e_0$ depends on the crystal structure in lattice dynamics and the nature of the physics under investigation. It is clear that a large range of values for the scale coefficient $e_0$ is possible due to different vibration frequencies. More works, especially experimental tests, are required to determine $e_0$ more accurately for CNTs. In the present work, we adopt the same scaling effect parameter $0 < e_0 < 1.5$ as used by Wang et al. [53] in our investigation of the small scale effect on the vibration behavior of CNTs quantitatively.

For the case of room or low temperature, we suppose $\alpha = -1.6 \times 10^{-6} \text{ K}^{-1}$ [37]. With the aspect ratio $L/d_1 = 60$ and the temperature change $\theta = 60 \text{ K}$, the relationship among the ratio $\chi_N$, the scale parameter $e_0a$ and $m$ is indicated in Fig. 2. With $m = 2$ and $\theta = 60 \text{ K}$, the relationship among the ratio $\chi_N$, the nonlocal parameter $e_0a$ and the aspect ratio $L/d_1$ is shown in Fig. 3. The ratio $\chi_N$ serves as an index to assess quantitatively the small scale effect on DWCNT buckling solution. It is clearly seen from Figs. 2 and 3 that the ratios $\chi_N$ is less than unity. This means that the application of the local Euler–Bernoulli beam model for CNT analysis would lead to an overprediction of the critical axial buckling strain if the small length scale effect between the individual carbon atoms in CNTs is neglected. As the scale parameter $e_0a$ increases, the ratio $\chi_N$ obtained for the nonlocal beam theory become smaller than those for its local counterpart. This reduction is especially significant for higher values of $m$, and thus the small scale effect cannot be neglected. The reduction may be explained as follows. The small scale effect makes the CNTs more flexible as the nonlocal model may be viewed as atoms linked by elastic springs [51] while the local continuum model assumes the spring constant to take on an infinite value. In sum, the nonlocal beam theory should be used if one needs accurate predictions of critical axial buckling strain of carbon nanotubes.

For the case of high temperature, we suppose $\alpha = 1.1 \times 10^{-8} \text{ K}^{-1}$ [37]. With the aspect ratio $L/d_1 = 60$ and the temperature change $\theta = 60 \text{ K}$, the variation of the ratio $\chi_N$ with the scale parameter $e_0a$ for various $m$ is shown in Fig. 4. With $m = 2$ and $\theta = 60 \text{ K}$, the variation of the ratio $\chi_N$ with the nonlocal parameter $e_0a$ for various aspect ratio $L/d_1$ is shown in Fig. 5. It is seen from Figs. 4 and 5 that the column buckling strain for the DWNT is related to the nonlocal parameter $e_0a$. It can be seen from

![Fig. 1. Dispersion curves of one-dimensional lattice [52.]](image-url)
Figs. 4 and 5 that the ratio \( \chi_N \) is less than unity, which is similar to the case of room or low temperature. Hence the critical axial buckling strain of the nanotubes based on the classical beam theory is over estimated.

It is expected that the small scale effect will diminish for a very slender CNT. Comparing its magnitude with the length of the slender tube, the small scale (the internal characteristic length) is so small that it can be regarded as zero. Figs. 3 and 5 confirm this point for both cases of low and high temperatures by showing the different variations of the ratio \( \chi_N \) versus the...
aspect ratio $L/d_1$ (length-to-diameter ratios) for $m = 2$ with different small scale parameter $e_0a$. Therefore, it is clear that the small scale effect is significant for short CNTs.

4.2. Discussion on temperature change effect

For the case of room or low temperature, we suppose $\alpha = -1.6 \times 10^{-6}$ K$^{-1}$ [37]. With the aspect ratio $L/d_1 = 60$ and the scale parameter $e_0a = 2$ nm, the relationship among the ratio $\chi_{th}$, the temperature change $\theta$ and $m$ is indicated in Fig. 6. With $m = 2$ and $e_0a = 2$ nm, the relationship among the ratio $\chi_{th}$, the temperature change $\theta$ and the aspect ratio $L/d_1$ is shown in Fig. 7. It is seen in from Figs. 6 and 7 that the column buckling strain for the DWNT is dependent on the temperature change $\theta$. The thermal effect on the buckling strain becomes more significant with the increase of the temperature change $\theta$ and the aspect ratio $L/d_1$ and becomes less significant with the increase of $m$. For instance, in the case $L/d_1 = 60$, we observe from Fig. 6 that the effect of temperature change on the buckling strain can be neglected when $m \geq 8$. It is also seen from Figs. 6 and 7 that the ratio $\chi_{th}$ is more than unity. This means that the buckling strain including the thermal effect is larger than that without considering the change of temperature and increases with the increase of temperature change.

For the case of high temperature, we suppose $\alpha = 1.1 \times 10^{-6}$ K$^{-1}$ [37]. With the aspect ratio $L/d_1 = 60$ and the scale parameter $e_0a = 2$ nm, the variation of the ratio $\chi_{th}$ with the temperature change $\theta$ for various $m$ is shown in Fig. 8. With $m = 2$ and $e_0a = 2$ nm, the variation of the ratio $\chi_{th}$ with the temperature change $\theta$ for various aspect ratio $L/d_1$ is shown in Fig. 9. It is seen from Figs. 8 and 9 that the column buckling strain for the DWNT is related to the temperature change $\theta$. Contrary to the case of room or low temperature, it can be seen from Figs. 8 and 9 that the ratio $\chi_{th}$ is less than unity. This means that the buckling strain including the thermal effect is smaller than those excluding the influence of temperature change. The thermal effect on the buckling strain increases with the increase of the temperature change $\theta$ and the aspect ratio $L/d_1$ and decreases with the increase of $m$. This is consistent with the case of low or room temperature. On the other hand, in the case of $L/d_1 = 60$, it can be seen from Fig. 8 that the influence of temperature change on the buckling strain can be ignored when $m \geq 8$, which is also consistent with the case of low or room temperature.
5. Conclusions

The buckling behavior of MWNTs with large aspect ratios under axial compression coupling with temperature change, incorporating a constitutive law that includes a length scale, has been analyzed using nonlocal Euler–Bernoulli beam model. In particular, an explicit expression is obtained for the critical buckling strain for a double-walled carbon nanotube. Incorporation of small length scale effects and temperature change are found to significantly affect the buckling behavior. From the results presented herein, it can be clearly seen that the small scale effect reduces the critical buckling strain. The influence of temperature change on the buckling strain of the double-walled carbon nanotube is also discussed. It is found that

Fig. 7. Thermal effects on the critical buckling strain of DWNT for various aspect ratio $L/d_1$ in the case of low or room temperature.

Fig. 8. Thermal effects on the critical buckling strain of DWNT for various $k$ in the case of high temperature.

Fig. 9. Thermal effects on the critical buckling strain of DWNT for various aspect ratio $L/d_1$ of high temperature.
the thermal effect on the buckling strain is dependent on the temperature changes, the aspect ratios, and the buckling modes of carbon nanotubes. It is hoped that the analytical buckling solutions presented herein will be useful for research work on nanostructures.

References