On the use of polynomial series with the Rayleigh–Ritz method

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Extensive literature emphasising the use of special sets of polynomials for Rayleigh–Ritz solutions to plate problems exists. It is commonly suggested that convergence can be improved by the use of special sets of polynomials, which are often orthogonal over the plate domain. This paper shows analytically that the polynomial series chosen does not affect convergence directly. It is shown that the only effect of the choice of set (for given polynomial degree) is on the numerical stability of the solution. These assertions are supported by numerical results for bending and vibration of laminated composite plates. © 1998 Elsevier Science Ltd. All rights reserved.

INTRODUCTION

The Rayleigh–Ritz method has been widely used for the analysis of laminated plates. In this method, solutions are obtained by minimising the total potential energy with respect to the coefficients of an approximating series representing the displacement. The only restrictions on the series chosen is that it satisfies the geometric boundary conditions [1], is complete [1] and does not inherently violate the natural boundary conditions [2]. When these conditions are met, solutions converge to the exact solution as more terms in the series are retained. Different series types (e.g. trigonometric, hyperbolic, polynomial) yield different results for the same number of terms in the series and the efficiency of the solution will depend to some extent on the type of series chosen.

In the case of isotropic or especially orthotropic plate problems, trigonometric series yield simple solutions since the stiffness and mass matrices are diagonal. Difficulties arise, however, when trigonometric series are applied to problems involving boundary conditions other than simple support or to generally orthotropic laminates. In these cases other types of series have been applied. Polynomial series, which allow straightforward algebraic manipulation, have recently been extensively used. Prior to the mid-1980s their use was limited because of the perceived difficulty in setting up series inherently satisfying the geometric boundary conditions. A means to overcome this difficulty by recursively generating higher terms of the series, based on starting polynomials specific to the boundary conditions under consideration, was presented by Bhat [3]. Bhat used the Gram–Schmit process to generate orthogonal sets of polynomials but reasons for selection of orthogonal rather than non-orthogonal polynomials were not given. Bhat compared his results for isotropic plates with those generated using the beam characteristic functions [4] and the simply supported plate functions [5]. Both types of functions consist of combinations of trigonometric and hyperbolic terms. It was shown that the orthogonal polynomials offered improved convergence.

Although Bhat did not suggest that the improved convergence was a direct result of the use of orthogonal functions, this would appear to be the interpretation of many researchers who have applied Bhat's orthogonal polynomials to other problems. It would appear that, in most instances where Bhat's approach has been used, emphasis is placed on construc-
tion of the orthogonal polynomials rather than on the ease by which starting polynomials can be generated for different boundary conditions [6–12]. Moreover, some recent results [13] suggest that the actual set of polynomials used plays a direct role in the convergence of the method. The objective of the present work is to show that the particular polynomial series used does not affect convergence directly. It will be shown that the results obtained for a given problem are strictly independent of the polynomial series chosen and will depend only on the degree of the polynomial represented by the series. The only practical difference between series representing the same degree polynomial will be shown to be the numerical stability (with respect to inversion and the extraction of eigenvalues) of the resulting stiffness and mass matrices. These assertions will be justified by presenting numerical results, obtained using a number of different sets of polynomials, for bending and vibration of laminated plates having different boundary conditions.

POLYNOMIAL BASES

Define a polynomial over \( \mathbb{R}^2 \) of degree \( n \) as any map

\[
P : \mathbb{R}^2 \to \mathbb{R}
\]

\[
\quad : (x,y) \mapsto \sum_{q=0}^{n} \sum_{i=0}^{q} c_{iq} x^i y^{q-i}
\]

where \( c_{iq} \in \mathbb{R} \). For convenience this polynomial can be written in matrix form. Define

\[
j(i,q) : (x,y) \mapsto x^i y^{q-i} c_{j(i,q)} = c_{iq}
\]

where \( 0 \leq q < n \) and \( 0 \leq i < q \), using the renumbering transform

\[
j(i,q) \mapsto q(q+1)/2 + 1 + i
\]

Define the column matrices

\[
C = [c_i]
\]

\[
T(x,y) = [t_j(x,y)]
\]

where \( i = 1, \ldots, k(n) \) and the matrix rank \( k(n) = (n + 1)(n + 2)/2 \). Then

\[
P : \mathbb{R}^2 \to \mathbb{R}
\]

\[
\quad : (x,y) \mapsto C^T T(x,y)
\]

Let \( [\mathbb{R}]_m \) be the set of all real column matrices of rank \( m \). The set of all polynomials over \( \mathbb{R}^2 \) of degree \( n \) is

\[
\mathcal{P}^n = \{ P = (C')^T T \cdot C' \in [\mathbb{R}]_m \}
\]

A complete basis for \( \mathcal{P}^n \) is a column vector \( \mathcal{B} = [B_1, \ldots, B_m] \), where each \( B_i \in \mathcal{P}^n \), such that there exists \( A_i \in [\mathbb{R}]_m \) such that

\[
(A_i)^T \mathcal{B} = P_i
\]

for every \( P_i \in \mathcal{P}^n \), subject to the linear independence condition that no real matrix \( A' \) besides the identity matrix exists such that

\[
A' \mathcal{B} = \mathcal{B}
\]

If there are two complete bases \( \mathcal{B} \) and \( \mathcal{B}' \) for \( \mathcal{P}^n \), then since \( \mathcal{B} \subseteq \mathcal{P}^n \) it follows from eqns that there exists a real matrix \( A \) such that

\[
A \mathcal{B}' = \mathcal{B}
\]

Hence one complete basis for \( \mathcal{P}^n \) can always be written in terms of any other.

Note too that it is obvious that \( \mathcal{B} = T \) is a complete basis for \( \mathcal{P}^n \) since in this case the various \( A_i \) are just the coefficients \( C' \) of the polynomials. Term this basis the trivial basis for \( \mathcal{P}^n \).

Given any complete basis \( \mathcal{B} \) for \( \mathcal{P}^n \), from eqns there exist real matrices \( A \) and \( A' \) such that

\[
A \mathcal{B} = \mathcal{B}'
\]

\[
\mathcal{B}' = A' \mathcal{B}
\]

implying that \( A' = A^{-1} \) and hence that \( A \) and \( A' \) are both square. Hence all complete bases for \( \mathcal{P}^n \) have the same number of elements. Consequently, since all polynomial series descriptions of the displacement must be complete bases no gains in efficiency can be expected by changing from one series to another.

For plates the total potential energy can be expressed as

\[
\Pi = \frac{1}{2} Q^T K Q - Q^T L
\]
for the two problems, respectively. A change of basis from $\mathcal{B}$ to $\mathcal{B}'$, where $A\mathcal{B} = \mathcal{B}'$, allows eqn (11) to be written as

$$
\Pi = \frac{1}{2} \mathcal{Q}'A'TK\mathcal{A}' - \mathcal{Q}'A'TL
$$

and

$$
\Pi = \frac{1}{2} \tilde{\mathcal{Q}}'A'TKA\tilde{\mathcal{A}} - \frac{1}{2} \lambda \tilde{\mathcal{Q}}'A'TMA\tilde{\mathcal{A}}
$$

(13)

Minimisation of this expression with respect to $\tilde{\mathcal{Q}}$ yields equations of the same form as eqn (12):

$$
\tilde{K}\tilde{Q} = \tilde{L} \text{ and } \tilde{K}Q = \lambda \tilde{M}Q
$$

but with transformed matrices $\tilde{K} = A'TKA\tilde{L} = A'TL$ and $\tilde{M} = A'TMA$. Hence a change of basis has no effect on the form of the problem but results in a new set of matrices which in general will have different conditioning to the set of matrices obtained using the original basis.

**NUMERICAL RESULTS AND DISCUSSION**

Perhaps the most widely used polynomials in the Rayleigh–Ritz method are the orthogonal series originally developed by Bhat [3], see for example [6–12]. An alternative approach to the generation of orthogonal polynomials was developed by Chow et al. [14] Several authors have proposed the use of simple polynomial series with no requirement for orthogonality. Barharlou & Leissa [15] proposed that the trivial basis be used in conjunction with constraint matrices. Qatu [16] also used non-orthogonal polynomial series but imposed boundary conditions using starting polynomials similar to those of Bhat. Recently Al-Obeid and Cooper [13] presented results using a similar approach to that of Qatu, differing only in the choice of dimensionless co-ordinates.

Results are presented here for symmetrically laminated plates with two different sets of boundary conditions: all edges simply supported and all edges clamped. Two laminate constructions are considered, namely $[0]_p$ and $[+45]_p$. The $[0]_p$ laminate is specially orthotropic and gives rapid convergence. Convergence for the $[+45]_p$ laminate is slower. The case of bending under a uniform transverse pressure is chosen for the load–deformation problem and the free vibration case is chosen for the eigenvalue problem.

The plates treated here have aspect ratio of unity and are constructed using a unidirectional CFRP with elastic properties [17] $E_1 = 138 \text{ GPa}$, $E_2 = 8.96 \text{ GPa}$, $G_{12} = 7.1 \text{ GPa}$ and $\nu_{12} = 0.3$. Through thickness shear deformation and rotary inertia are ignored.

To allow identification of possible differences in the convergence of the various approaches, results were generated for increasing polynomial degree. For the simply supported plates, series having 64, 81 and 100 terms are used while for the clamped plates series having 100, 121 and 144 terms are used. Higher degree series are used for the clamped plates since the additional geometric boundary conditions reduce the number of independent coefficients in a series of given degree.

Values obtained using polynomial approaches given in Refs [13–16] are compared with those obtained using Bhat's polynomials. Numerical calculations were all performed using Matlab Version 3.51 on a 486-DX50 PC.

The dimensionless parameters

$$
\delta = \frac{\delta E_1 h^3}{12(1-\nu_{12}\nu_{21})a^4q}
$$

and

$$
\Omega = \frac{\omega a^2}{h} \sqrt{\frac{12(1-\nu_{12}\nu_{21})p}{E_1}}
$$

(15)

are used for deflection and frequency respectively. ($\delta$ is the deflection, $E_1$ is the lamina longitudinal Young's modulus, $h$ is the plate thickness, $\nu_{12}$ and $\nu_{21}$ are the lamina major and minor Poisson's ratios, $a$ is the plate length, $q$ is the transverse pressure, $\rho$ is the material density and $\omega$ is the natural frequency.)

Tables 1 and 2 show the results obtained using Bhat's polynomials for the simply supported and clamped plates. $\omega_c$ is the deflection parameter at the plate centre and $w_q$ is the same parameter at the quarter point (i.e. $x$ and $y$ both one quarter of length). Subscripts to $\Omega$ indicate the mode of vibration.

Table 3 shows the minimum and maximum number of significant digits to which agreement with the above results is obtained when using various other sets of polynomials. The good agreement between the results of Refs [14–16]
Table 1. Results for simply supported plates (Bhat's polynomials)

<table>
<thead>
<tr>
<th></th>
<th>[0°], plate</th>
<th></th>
<th>[+45°], plate</th>
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</thead>
<tbody>
<tr>
<td>Number of terms</td>
<td></td>
<td>Number of terms</td>
<td></td>
</tr>
<tr>
<td></td>
<td>64</td>
<td>81</td>
<td>100</td>
</tr>
<tr>
<td>(w_c \times 10^2)</td>
<td>1.2059626</td>
<td>1.2070669</td>
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<tr>
<td>(w_q \times 10^3)</td>
<td>6.691837</td>
<td>6.6831290</td>
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<tr>
<td>(\Omega_1 \times 10^4)</td>
<td>1.1289718</td>
<td>1.1289717</td>
<td>1.1289717</td>
</tr>
<tr>
<td>(\Omega_2 \times 10^5)</td>
<td>1.7131899</td>
<td>1.7131898</td>
<td>1.7131782</td>
</tr>
<tr>
<td>(\Omega_3 \times 10^6)</td>
<td>2.9020899</td>
<td>2.8696301</td>
<td>2.8696301</td>
</tr>
<tr>
<td>(\Omega_4 \times 10^7)</td>
<td>4.0740931</td>
<td>4.0740931</td>
<td>4.0740231</td>
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Table 2. Results for clamped plates (Bhat's polynomials)

<table>
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<tr>
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<th>[0°], plate</th>
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<th>[+45°], plate</th>
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<tbody>
<tr>
<td>Number of terms</td>
<td></td>
<td>Number of terms</td>
<td></td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>121</td>
<td>144</td>
</tr>
<tr>
<td>(w_c \times 10^3)</td>
<td>2.6739550</td>
<td>2.6753911</td>
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<tr>
<td>(w_q \times 10^3)</td>
<td>1.1584915</td>
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<tr>
<td>(\Omega_1 \times 10^4)</td>
<td>2.3853209</td>
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<tr>
<td>(\Omega_2 \times 10^5)</td>
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<tr>
<td>(\Omega_3 \times 10^6)</td>
<td>4.1768453</td>
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<tr>
<td>(\Omega_4 \times 10^7)</td>
<td>6.0235888</td>
<td>5.9975638</td>
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Table 3. Number of significant digits in agreement

<table>
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<tbody>
<tr>
<td></td>
<td>min</td>
<td>max</td>
<td>min</td>
<td>max</td>
</tr>
<tr>
<td>[0°], plate simply supported</td>
<td>(w_c)</td>
<td>15</td>
<td>16</td>
<td>13</td>
</tr>
<tr>
<td></td>
<td>(w_q)</td>
<td>15</td>
<td>15</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>(\Omega_{1-4})</td>
<td>13</td>
<td>16</td>
<td>12</td>
</tr>
<tr>
<td>[+45°], plate simply supported</td>
<td>(w_c)</td>
<td>13</td>
<td>14</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>(w_q)</td>
<td>13</td>
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<td>10</td>
</tr>
<tr>
<td></td>
<td>(\Omega_{1-4})</td>
<td>13</td>
<td>16</td>
<td>11</td>
</tr>
<tr>
<td>[0°], plate clamped</td>
<td>(w_c)</td>
<td>14</td>
<td>16</td>
<td>11</td>
</tr>
<tr>
<td></td>
<td>(w_q)</td>
<td>14</td>
<td>14</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>(\Omega_{1-4})</td>
<td>14</td>
<td>16</td>
<td>12</td>
</tr>
<tr>
<td>[+45°], plate clamped</td>
<td>(w_c)</td>
<td>11</td>
<td>14</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>(w_q)</td>
<td>11</td>
<td>13</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>(\Omega_{1-4})</td>
<td>11</td>
<td>15</td>
<td>8</td>
</tr>
</tbody>
</table>
approaches of Refs [3] and [14–16], the condition numbers are mostly within one order of magnitude and all tend to decrease as the number of terms in the series increases. Of these approaches, the smallest condition numbers occur with the method of Barharlou and Leissa [15] when applied to the [+45], clamped plates where the minimum condition of the stiffness matrix is found to be of order $10^{-12}$ and of the mass matrix of order $10^{-14}$. The corresponding smallest condition numbers for the three remaining approaches are only of the order $10^{-9}$ and $10^{-12}$, respectively. This suggests that the poorer agreement of the results of Ref. [15] for the [+45], clamped plate is a consequence of poor matrix conditioning. However, it must be borne in mind that some numerical error can be introduced in this method during the generation of the constraint matrices which, in itself, involves matrix inversions. Hence it is likely that Bhat’s [3] method of enforcing boundary conditions will give rise to solutions which are less sensitive to numerical instability.

Ill-conditioning of the stiffness and mass matrices explains why the results of the method of Al-Obeid and Cooper [13] do not agree well with those of the other approaches. Here the matrices tend to have poor condition numbers for all of the plates analysed, with the minimum values for both the stiffness and mass matrices being of order $10^{-16}$. The results are thus likely to be heavily dependent on the eigenvalue extraction algorithm and the type of computer used. This could explain why the values presented by Al-Obeid & Cooper [13] are, for similar polynomial degree, quite different to those of Chow et al. [14] (and indeed to those generated by the present authors using the method of Ref. [13]). Al-Obeid and Cooper’s [13] conclusion that the polynomials developed yield superior convergence compared with polynomials developed by other authors is thus most likely based on unreliable numerical data.

CONCLUDING REMARKS

It has been shown that the particular set of polynomials used does not strictly affect results obtained by the Rayleigh–Ritz method. Different sets of polynomials of the same degree are observed to have no effect the form of the problem but may be expected to influence the conditioning of the stiffness, mass and geometric stiffness matrices. Thus the argument that convergence can be improved by the selection of special sets of polynomials is erroneous. In particular, the emphasis placed on the generation of orthogonal polynomials should be questioned in light of the present work.

The results clearly show that, from a designer’s point of view, all of the polynomials considered offer adequate performance. The only real need for the use of special polynomials might be to ensure numerical stability for cases where higher degree polynomials are necessary. Studies focussing on the relationship between polynomial series construction and matrix stability could thus be justified. It is worth noting that orthogonal polynomials will probably not be best in this regard for plates since, for numerical stability of the relevant matrices obtained by the Rayleigh–Ritz method, one would be seeking orthogonality on the level of the second derivatives of the functions and not of the functions themselves.

REFERENCES


